Unit F1

Limits

Introduction to Book F

In this book we put the foundations of calculus on a firm logical basis. At the end of Book D *Analysis 1* you met the idea of a continuous function, and at the start of this book you will meet the related ideas of a *limit* of a function and of *uniform continuity*. Then we use these ideas to study *differentiation* and *integration* in detail. Finally, we discuss the representation of functions by *power series*.

You will meet many applications of these ideas, including:

- a technique for proving inequalities such as

$$\log(1+x) > x - \tfrac{1}{2}x^2, \quad \text{for } x \in (0, \infty)$$

- a function which is continuous but nowhere differentiable
- Stirling's approximate formula for $n!$
- several remarkable exact formulas for π, including Wallis' Product,

$$\frac{\pi}{2} = \lim_{n\to\infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \cdots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right)$$

- an elegant proof that π is irrational.

Introduction

In this unit you will meet the concept of a *limit of a real function*, which is closely related to the idea of a continuous function. Roughly speaking, a real function f has a limit at a point c if it is either continuous at c, or if it is defined near c and we can assign a value to $f(c)$ that makes the function continuous at c. You will also study various types of asymptotic behaviour of functions – that is, their behaviour when the domain variable or the codomain variable becomes arbitrarily large. For example, you will see that if $n \in \mathbb{N}$, then $x^n/e^x \to 0$ as $x \to \infty$.

In the second half of the unit you will return to the topic of continuity and study an alternative definition, the so-called ε-δ definition of continuity. This is equivalent to the definition based on sequences that you studied in Unit D4 *Continuity* and, although it may appear to be more abstract, it is easier to use in certain situations. You will meet several unusual functions and see how the two definitions of continuity can be used to investigate at which points they are continuous or discontinuous. You may be surprised at the results!

Finally, you will meet the concept of *uniform continuity*. This is a stronger form of continuity, defined using the ε-δ approach, and it will play an important role when you study the integration of continuous functions.

Many of the results in Sections 1 and 2 are analogues of results on sequences and continuity covered in previous units, so we omit their proofs.

1 Limits of functions

In this section you will study the concept of a limit of a real function (that is, a function whose domain and codomain are subsets of $\mathbb{R}$). The notion of a limit is of fundamental importance to differentiation, which you will study in the next unit, so you should make sure you have a good understanding of this material.

1.1 What is a limit of a function?

Sometimes we need to understand the behaviour of a function that is defined near a particular point, but not at the point itself. For example, consider the function

$$f(x) = \frac{\sin x}{x} \quad (x \in \mathbb{R} - \{0\}).$$

(This function arises when we prove that the sine function is differentiable, as you will see in Unit F2 *Differentiation*.) The graph of f in Figure 1 suggests that as x gets closer and closer to 0, $f(x)$ takes values which are closer and closer to 1.

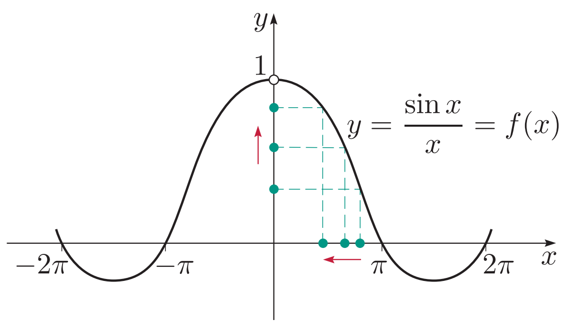


Figure 1 The graph of $y = \dfrac{\sin x}{x}$

On the other hand, consider the function

$$g(x) = \sin\frac{1}{x} \quad (x \in \mathbb{R} - \{0\}).$$

In this case, when x takes values close to 0, the values taken by $g(x)$ do not lie close to any *single* real number: as you can see in Figure 2, g is highly oscillatory near 0.

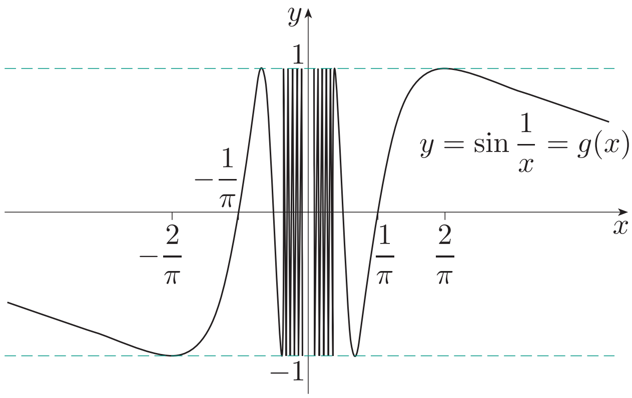


Figure 2 The graph of $y = \sin\frac{1}{x}$

We say that the function f has a *limit* as x tends to 0, but the function g does not. We now make this concept precise.

First we introduce the idea of a **punctured neighbourhood** of a point c. This is simply a bounded open interval with midpoint c, from which the point c itself has been removed. We use the notation $N_r(c)$ for a punctured neighbourhood of length $2r$ with centre c, so

$$N_r(c) = (c - r, c) \cup (c, c + r), \quad \text{where } r > 0,$$

as illustrated in Figure 3. For example $N_1(3) = (2, 3) \cup (3, 4)$.

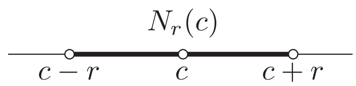


Figure 3 The punctured neighbourhood $N_r(c)$

Definition

Let f be a function defined on a punctured neighbourhood $N_r(c)$ of c. Then $f(x)$ **tends to the limit l as x tends to c** if $l \in \mathbb{R}$ and

for each sequence (x_n) in $N_r(c)$ such that $x_n \to c$,
$f(x_n) \to l$.

In this case, we write

$$\lim_{x\to c} f(x) = l \quad \text{or} \quad f(x) \to l \text{ as } x \to c.$$

Remarks

1. The definition applies whether or not the function f is defined at the point c and, if it is defined there, irrespective of the value of $f(c)$.
2. Note that the limit $\lim_{x\to c} f(x)$, if it exists, does not depend on which punctured neighbourhood of c is considered (that is, it does not depend on r). It is also important to note that the limit must be the same for *every* possible sequence (x_n).

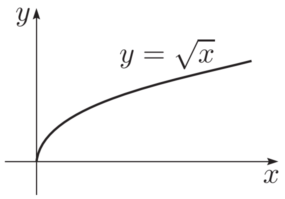


Figure 4 The graph of $y = \sqrt{x}$

3. The above definition does not allow us to state that $\lim_{x \to 0} \sqrt{x}$ exists, because the domain $[0, \infty)$ of $f(x) = \sqrt{x}$ does not contain any punctured neighbourhood of 0. Later in this section we introduce the idea of a *one-sided limit*, and see that $f(x) = \sqrt{x}$ has limit 0 as x tends to 0 *from the right*. The graph of this function is shown in Figure 4.
4. Because this definition of a limit involves sequences, we often use results about sequences to determine whether a function has a limit at a particular point.

We now use this definition to prove that $(\sin x)/x \to 1$ as $x \to 0$, as we guessed earlier.

Theorem F1

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Proof To prove this result we first use two trigonometric inequalities to establish upper and lower bounds for $(\sin x)/x$ on a punctured neighbourhood of 0. This then enables us to apply the Squeeze Rule for convergent sequences from Subsection 3.3 of Unit D2 *Sequences*.

First note that the function $x \longmapsto (\sin x)/x$ is defined on every punctured neighbourhood of 0.

We now use the inequality

$$\sin x \leq x, \quad \text{for } 0 < x \leq \pi/2,$$

(proved in Subsection 2.3 of Unit D4) to deduce that

$$\frac{\sin x}{x} \leq 1, \quad \text{for } 0 < x \leq \pi/2. \tag{1}$$

Next we require the inequality

$$x \leq \tan x, \quad \text{for } 0 < x < \pi/2, \tag{2}$$

which follows by comparing the area of a sector of a disc of radius 1 (shown in Figure 5(a)) with that of a certain right-angled triangle which contains the sector (shown in Figure 5(b)).

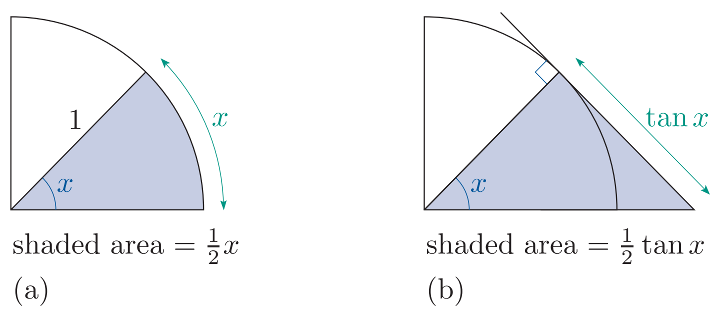


Figure 5 (a) A sector of a disc (b) A triangle containing the sector

Recall that a sector of angle θ in a disc of radius r has area $\frac{1}{2}\theta r^2$.

Since $\cos x > 0$ for $0 < x < \pi/2$, we deduce from inequality (2) that

$$\cos x \leq \frac{\sin x}{x}, \quad \text{for } 0 < x < \pi/2.$$

Remember that a real function f is *even* if $f(x) = f(-x)$ for $x \in \mathbb{R}$.

Thus, by inequality (1) and the fact that the functions $x \longmapsto \cos x$ and $x \longmapsto (\sin x)/x$ are both even,

$$\cos x \leq \frac{\sin x}{x} \leq 1, \quad \text{for } 0 < |x| < \pi/2. \tag{3}$$

We have now established lower and upper bounds for $(\sin x)/x$ in the punctured neighbourhood $N_{\pi/2}(0)$.

Now suppose that (x_n) is any null sequence in the punctured neighbourhood $N_{\pi/2}(0)$. Then

$$\cos x_n \leq \frac{\sin x_n}{x_n} \leq 1, \quad \text{for } n = 1, 2, \ldots, \tag{4}$$

by inequalities (3). Since $x_n \to 0$, we have $\cos x_n \to 1$, because the cosine function is continuous at 0 and $\cos 0 = 1$. Hence, by inequalities (4) and the Squeeze Rule for sequences,

$$\frac{\sin x_n}{x_n} \to 1 \text{ as } n \to \infty,$$

as required. ■

The limit in Theorem F1 was quite tricky to establish, but usually there are simpler ways to find limits. For example, we can determine many limits of functions by using the Combination Rules for sequences which you met in Unit D2.

Worked Exercise F1

Prove that each of the following functions tends to a limit as x tends to 2, and determine these limits.

(a) $f(x) = \dfrac{x^2 - 4}{x - 2}$ (b) $f(x) = \dfrac{x^3 - 3x - 2}{x^2 - 3x + 2}$

Solution

(a) The domain of f is $\mathbb{R} - \{2\}$, so f is defined on each punctured neighbourhood of 2. Also,

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x - 2} = x + 2, \quad \text{for } x \neq 2.$$

We can cancel $x - 2$, since $x \neq 2$.

Thus if (x_n) is any sequence in $\mathbb{R} - \{2\}$ such that $x_n \to 2$, then

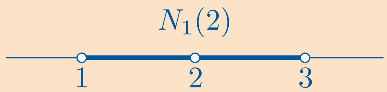
$$f(x_n) = x_n + 2 \to 2 + 2 = 4 \quad \text{as } n \to \infty,$$

by the Sum Rule for sequences. Hence

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$$

(b) Since $x^2 - 3x + 2 = (x-2)(x-1)$, the domain of f is $\mathbb{R} - \{1, 2\}$.

Because f is not defined at 1, the largest punctured neighbourhood of 2 in which f is defined is $N_1(2)$, illustrated below.



Thus f is defined on $N_1(2)$ and

$$f(x) = \frac{x^3 - 3x - 2}{x^2 - 3x + 2} = \frac{(x-2)(x^2 + 2x + 1)}{(x-2)(x-1)} = \frac{x^2 + 2x + 1}{x - 1},$$

for $x \in N_1(2)$. Thus if (x_n) lies in $N_1(2)$ and $x_n \to 2$, then

$$f(x_n) = \frac{x_n^2 + 2x_n + 1}{x_n - 1} \to \frac{4 + 4 + 1}{2 - 1} = 9,$$

by the Combination Rules for sequences. Hence

$$\lim_{x \to 2} \frac{x^3 - 3x - 2}{x^2 - 3x + 2} = 9.$$

Later in this section you will meet further techniques for finding limits. First, however, we give a strategy for proving that a limit does *not* exist.

Strategy F1

Let f be a real function defined on a punctured neighbourhood $N_r(c)$ of c.

To show that $\lim_{x \to c} f(x)$ does not exist, either

- find two sequences (x_n) and (y_n) in $N_r(c)$ which tend to c, such that $(f(x_n))$ and $(f(y_n))$ have different limits, or
- find a sequence (x_n) in $N_r(c)$ which tends to c such that $f(x_n) \to \infty$ or $f(x_n) \to -\infty$.

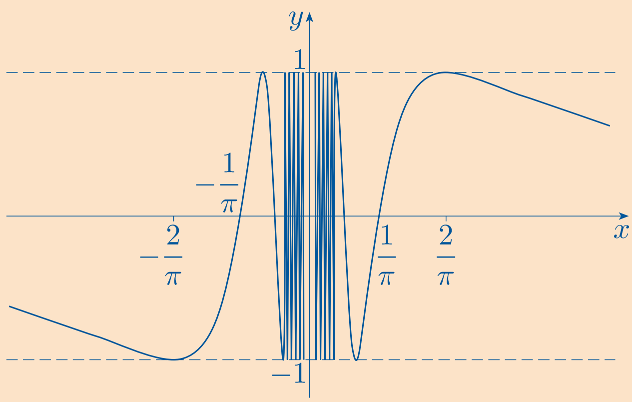
Worked Exercise F2

Prove that each of the following functions does not tend to a limit as x tends to 0.

(a) $f(x) = \sin(1/x)$ (b) $f(x) = 1/x$

Solution

(a) The graph of f is shown below.



The graph suggests that we should be able to use the first part of Strategy F1, with one sequence of points whose images under f are equal to 1 and one sequence of points whose images under f are equal to -1.

The function $f(x) = \sin(1/x)$ has domain $\mathbb{R} - \{0\}$. To prove that $f(x)$ does not tend to a limit as x tends to 0, we choose two null sequences (x_n) and (y_n) in $\mathbb{R} - \{0\}$ such that

$$f(x_n) \to 1 \quad \text{whereas} \quad f(y_n) \to -1.$$

To do this we use the facts that

$$\sin(2n\pi + \pi/2) = 1 \quad \text{and} \quad \sin(2n\pi + 3\pi/2) = -1, \quad \text{for } n \in \mathbb{Z}.$$

It follows that if we choose

$$x_n = \frac{1}{2n\pi + \pi/2} \quad \text{and} \quad y_n = \frac{1}{2n\pi + 3\pi/2}, \quad n = 1, 2, \ldots,$$

then $x_n \to 0$, $y_n \to 0$ and, for $n = 1, 2, \ldots$,

$$f(x_n) = \sin(1/x_n) = 1 \quad \text{and} \quad f(y_n) = \sin(1/y_n) = -1.$$

So

$$f(x_n) \to 1 \text{ as } n \to \infty \quad \text{and} \quad f(y_n) \to -1 \text{ as } n \to \infty.$$

Hence $f(x) = \sin(1/x)$ does not tend to a limit as x tends to 0.

(b) The behaviour of $f(x)$ as $x \to 0$ suggests that we can use the second part of Strategy F1.

The function $f(x) = 1/x$ has domain $\mathbb{R} - \{0\}$. The sequence $(1/n)$ lies in $\mathbb{R} - \{0\}$ and tends to 0, but

$$f(1/n) = \frac{1}{1/n} = n \to \infty \quad \text{as } n \to \infty.$$

Hence $f(x) = 1/x$ does not tend to a limit as x tends to 0.

Here are some limits of functions for you to consider. Recall that $\lfloor x \rfloor$ is the integer part of x.

Exercise F1

Determine whether each of the following limits exists, and evaluate those limits which do exist.

(a) $\displaystyle\lim_{x\to 0} \frac{x^2+x}{x}$ (b) $\displaystyle\lim_{x\to 1} \lfloor x \rfloor$ (c) $\displaystyle\lim_{x\to 0} \log|x|$

1.2 Limits and continuity

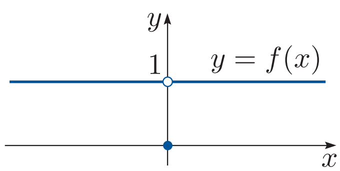


Figure 6 The graph of a function which takes the value 1 except at $x = 0$.

Consider the function

$$f(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

whose graph is shown in Figure 6. Does this function tend to a limit as x tends to 0 and, if so, what is the limit? Well, if (x_n) is any null sequence with non-zero terms, then

$$f(x_n) = 1, \quad \text{for } n = 1, 2, \ldots,$$

so

$$f(x_n) \to 1 \quad \text{as } n \to \infty.$$

Hence

$$\lim_{x\to 0} f(x) = 1.$$

This example illustrates the fact that the value of a limit $\displaystyle\lim_{x\to c} f(x)$ is not affected by the value of $f(c)$, if f happens to be defined at c.

However, the following theorem shows that if f is defined and *continuous* at c, then the value of the limit must be $f(c)$, and the converse statement is also true.

Theorem F2

Let f be a function defined on an open interval I, with $c \in I$. Then

f is continuous at c

if and only if

$$\lim_{x \to c} f(x) = f(c).$$

The proof of Theorem F2 uses the fact that, in this situation, the definition of continuity of f at c, from Subsection 2.1 of Unit D4, is almost identical to the definition of the existence of $\lim_{x \to c} f(x)$, with this limit equal to $f(c)$. The only difference is that, in the former case, we allow the terms of the sequences (x_n) which appear in the definition to equal c. We omit the details of this proof.

Theorem F2 makes it easy to calculate many limits of continuous functions. For example, to determine

$$\lim_{x \to 2}(3x^5 - 5x^2 + 1),$$

we use the fact that the function $f(x) = 3x^5 - 5x^2 + 1$ is continuous on $\mathbb{R}$, since f is a polynomial. Hence, by Theorem F2,

$$\lim_{x \to 2}(3x^5 - 5x^2 + 1) = f(2) = 77.$$

As a further example, you saw in Worked Exercise D48 in Unit D4 that the function

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is continuous at 0. Thus, by Theorem F2,

$$\lim_{x \to 0} x^2 \sin(1/x) = 0.$$

On the other hand, we saw in Worked Exercise F2(a) that

$$\lim_{x \to 0} \sin(1/x) \ \text{ does not exist.}$$

It follows from Theorem F2 that, no matter how we try to extend the domain of the function $f(x) = \sin(1/x)$ to include $x = 0$, we can never obtain a continuous function.

Exercise F2

Use Theorem F2 to determine the following limits.

(a) $\lim_{x \to 2} \sqrt{x}$ (b) $\lim_{x \to \pi/2} \sqrt{\sin x}$ (c) $\lim_{x \to 1} \dfrac{e^x}{1 + x}$

In the remainder of this unit we use Theorem F2 often, but we do not always refer to it explicitly.

1.3 Rules for limits

As you might expect from your experience with sequences, series and continuous functions, limits of functions can often be found by using various rules. First we state the Combination Rules for limits. These can be deduced from the corresponding rules for sequences; we omit the details.

Theorem F3 Combination Rules for limits

If $\lim_{x\to c} f(x) = l$ and $\lim_{x\to c} g(x) = m$, then:

Sum Rule $\lim_{x\to c}(f(x)+g(x)) = l+m$

Multiple Rule $\lim_{x\to c} \lambda f(x) = \lambda l$, for $\lambda \in \mathbb{R}$

Product Rule $\lim_{x\to c} f(x)g(x) = lm$

Quotient Rule $\lim_{x\to c} f(x)/g(x) = l/m$, provided that $m \neq 0$.

For example, since

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}(x^2+1) = 1,$$

we have, by the Combination Rules,

$$\lim_{x\to 0}\left(\frac{\sin x}{x} + 2(x^2+1)\right) = 1 + 2\times 1 = 3.$$

Next we discuss the composition of limits. For example, consider the behaviour of

$$\frac{\sin(x^2)}{x^2},$$

as x tends to 0. This function can be written in the form

$$\frac{\sin u}{u}, \quad \text{where } u = x^2.$$

Now

$$u = x^2 \to 0 \text{ as } x \to 0 \quad \text{and} \quad \frac{\sin u}{u} \to 1 \text{ as } u \to 0,$$

which suggests that

$$\frac{\sin(x^2)}{x^2} \to 1 \text{ as } x \to 0.$$

To justify this composition of limits, we use the following Composition Rule. This can be proved using properties of convergent sequences, limits and continuous functions that you have already met, but we do not give the details here.

Theorem F4 Composition Rule for limits

If $\lim_{x \to c} f(x) = l$ and $\lim_{x \to l} g(x) = L$, then

$$\lim_{x \to c} g(f(x)) = L,$$

provided that

either $f(x) \neq l$, for all x in some $N_r(c)$, where $r > 0$,

or g is defined at l and continuous at l.

Remarks

1. When using this rule, it is important to remember that the limit for g is as $x \to l$ and not as $x \to c$. This is perhaps easier to see if we rewrite the theorem as follows:

   If

   $$f(x) \to l \text{ as } x \to c$$

   and

   $$g(x) \to L \text{ as } x \to l,$$

   then

   $$g(f(x)) \to L \text{ as } x \to c.$$

2. Before you use the Composition Rule, you should check that one of the two provisos to Theorem F4 holds. (Sometimes both provisos will hold, but you only ever need to check that one does.)

Perhaps surprisingly, the Composition Rule is *false* if we omit both of the provisos. For example, if

$$f(x) = 1 \quad \text{and} \quad g(x) = \begin{cases} 2, & x \neq 1, \\ 0, & x = 1, \end{cases}$$

then

$$\lim_{x \to 0} f(x) = 1 \quad \text{and} \quad \lim_{x \to 1} g(x) = 2$$

so the Composition Rule would give $\lim_{x \to 0} g(f(x)) = 2$. However, because $f(x) = 1$ for all values of x and $g(1) = 0$, we actually have

$$\lim_{x \to 0} g(f(x)) = 0 \neq 2.$$

For this example, neither of the provisos to Theorem F4 holds, because there is no punctured neighbourhood of 0 in which $f(x) \neq 1$, and g is not continuous at 1.

However, in nearly all cases in practice and in all the examples you will meet in this module, at least one of the provisos holds (though you should check this).

This leads to the following strategy for using the Composition Rule.

Strategy F2

To use the Composition Rule to evaluate a limit of a function of the form $g(f(x))$ as $x \to c$, do the following.

1. Substitute $u = f(x)$ and show that, for some l,
$$u = f(x) \to l \;\text{ as } x \to c.$$
2. Show that, for some L,
$$g(u) \to L \;\text{ as } u \to l.$$
3. Check that one of the provisos holds.
4. Deduce that
$$g(f(x)) \to L \;\text{ as } x \to c.$$

The following worked exercise illustrates how to apply this strategy.

Worked Exercise F3

Determine the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x}$ (b) $\displaystyle\lim_{x\to 0}\left(1+\left(\frac{\sin x}{x}\right)^2\right)$

Solution

(a) We want to use Strategy F2 so we identify functions f and g such that $\dfrac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x} = g(f(x))$.

We can write
$$\frac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x} = g(f(x)), \quad \text{where } f(x) = \tfrac{1}{2}x \text{ and } g(x) = \frac{\sin x}{x}.$$

We now follow the steps of Strategy F2.

Substituting $u = f(x) = \frac{1}{2}x$, we have
$$u = \tfrac{1}{2}x \to u(0) = 0 \;\text{ as } x \to 0,$$
since u is continuous at 0, and
$$g(u) = \frac{\sin u}{u} \to 1 \;\text{ as } u \to 0.$$
The first proviso of the Composition Rule holds because $f(x) \neq 0$ for $x \in N_1(0)$, for example.

Notice that the second proviso does not hold in this case, since g is undefined at 0. However, we only need one of the provisos to hold to use the Composition Rule.

Thus, by the Composition Rule,

$$g(f(x)) = \frac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x} \to 1 \quad \text{as } x \to 0.$$

(b) We can write

$$1 + \left(\frac{\sin x}{x}\right)^2 = g(f(x)),$$

where

$$f(x) = \frac{\sin x}{x} \text{ and } g(x) = 1 + x^2.$$

Substituting $u = f(x) = \dfrac{\sin x}{x}$, we obtain

$$u = \frac{\sin x}{x} \to 1 \quad \text{as } x \to 0,$$
$$g(u) = 1 + u^2 \to 1 + 1 = 2 \quad \text{as } u \to 1,$$

since g is continuous at 1, which also tells us that the second proviso to the Composition Rule holds.

Alternatively you could have noted that the first proviso holds, since $f(x) \neq 1$ for $x \in N_1(0)$, for example.

Thus, by the Composition Rule,

$$g(f(x)) = 1 + \left(\frac{\sin x}{x}\right)^2 \to 2 \quad \text{as } x \to 0.$$

Exercise F3

Use the Combination Rules and the Composition Rule to determine the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{2x + x^2}$ (b) $\displaystyle\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x}$ (c) $\displaystyle\lim_{x\to 0} \left(\frac{x}{\sin x}\right)^{1/2}$

(d) $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x}$

Hint: In part (d), use the identity $\cos x = 1 - 2\sin^2\left(\frac{1}{2}x\right)$.

There is also a Squeeze Rule for limits, analogous to the Squeeze Rules for sequences and continuous functions, whose proof we omit. This is illustrated in Figure 7.

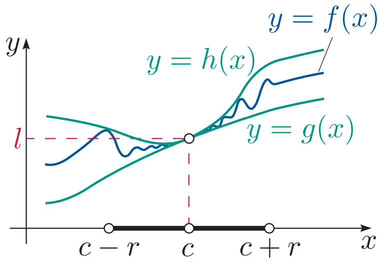


Figure 7 The Squeeze Rule for limits

Theorem F5 Squeeze Rule for limits

Let f, g and h be functions defined on $N_r(c)$, for some $r > 0$. If

(a) $g(x) \le f(x) \le h(x)$, for $x \in N_r(c)$

(b) $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = l$,

then

$$\lim_{x\to c} f(x) = l.$$

In the proof of Theorem F1, we showed that $\lim_{x\to 0} (\sin x)/x = 1$, using the inequalities

$$\cos x \le \frac{\sin x}{x} \le 1, \quad \text{for } 0 < |x| < \pi/2.$$

This was, in essence, an application of the Squeeze Rule for limits, with $f(x) = (\sin x)/x$, $g(x) = \cos x$ and $h(x) = 1$. Since $\lim_{x\to 0} \cos x = 1$, the result follows.

In the next exercise the Squeeze Rule is used to establish another important limit.

Exercise F4

(a) Use the inequalities

$$1 + x \le e^x \le \frac{1}{1-x}, \quad \text{for } |x| < 1,$$

(proved in Corollary D49 in Unit D4) to show that

$$1 - \frac{|x|}{1-x} \le \frac{e^x - 1}{x} \le 1 + \frac{|x|}{1-x}, \quad \text{for } 0 < |x| < 1.$$

(b) Deduce from part (a) that

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

The limit found in Exercise F4 is one of the basic limits we often use. Here we record three such limits for future reference. The first was proved in Theorem F1, the second in Exercise F3(d) and the third in Exercise F4(b).

Theorem F6 Three basic limits

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(b) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

(c) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

1.4 One-sided limits

Earlier we mentioned that $\lim_{x \to 0} \sqrt{x}$ is not defined because the function $f(x) = \sqrt{x}$ is not defined on any punctured neighbourhood of 0. However, this function does tend to 0 as x tends to 0 *from the right*.

Definitions

Let f be a function defined on $(c, c + r)$, for some $r > 0$. Then $f(x)$ **tends to the limit l as x tends to c from the right** if

for each sequence (x_n) in $(c, c + r)$ such that $x_n \to c$,
$f(x_n) \to l$.

In this case, we write

$$\lim_{x \to c^+} f(x) = l \quad \text{or} \quad f(x) \to l \ \text{ as } x \to c^+.$$

There is a similar definition for a limit **as x tends to c from the left**, in which $(c, c + r)$ is replaced by $(c - r, c)$. In this case, we write

$$\lim_{x \to c^-} f(x) = l \quad \text{or} \quad f(x) \to l \ \text{ as } x \to c^-.$$

We also refer to

$$\lim_{x \to c^+} f(x) \quad \text{and} \quad \lim_{x \to c^-} f(x)$$

as **right** and **left limits**, respectively.

Sometimes both right and left limits exist but are different, as you will see in the next worked exercise.

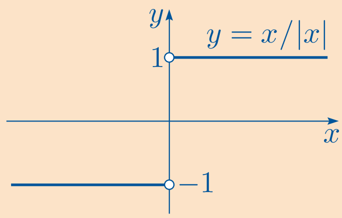
Worked Exercise F4

Prove that the following function tends to different limits as x tends to 0 from the right and from the left.

$$f(x) = \frac{x}{|x|} \quad (x \in \mathbb{R} - \{0\})$$

Solution

The graph of the function is shown below.



The function f is defined on $(0, 1)$ and $f(x) = 1$ on this open interval.

We could take any open interval of the form $(0, r)$ with $r > 0$ here.

Thus if (x_n) is a null sequence in $(0, 1)$, then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1.$$

Hence $\lim\limits_{x\to 0^+} f(x) = 1$.

Similarly, f is defined on $(-1, 0)$ and $f(x) = -1$ on this interval.

We could take any open interval of the form $(-r, 0)$ with $r > 0$ here.

Thus if (x_n) is a null sequence in $(-1, 0)$, then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} -1 = -1.$$

Hence $\lim\limits_{x\to 0^-} f(x) = -1$.

Since $-1 \neq 1$, the limits of $f(x)$ as x tends to 0 from the right and from the left are different.

The relationship between one-sided limits and ordinary limits is given by the following result, whose proof we omit.

Theorem F7

Let the function f be defined on $N_r(c)$, for some $r > 0$. Then

$$\lim_{x \to c} f(x) = l$$

if and only if

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = l.$$

Analogues of the Combination Rules, the Composition Rule (and Strategy F2) and the Squeeze Rule can also be used to determine one-sided limits. In the statements of these rules, we simply replace $\lim\limits_{x \to c}$ by $\lim\limits_{x \to c^+}$ or $\lim\limits_{x \to c^-}$, and replace $N_r(c)$ by $(c, c+r)$ or $(c-r, c)$, as appropriate. Also, Strategy F1 can be adapted to show that a one-sided limit does *not* exist; the sequences (x_n) and (y_n) must be chosen to tend to c from the right or from the left, as appropriate.

There is also a version of Theorem F2 for one-sided limits, as follows.

Theorem F8

Let f be a function whose domain is an interval I with a finite left-hand endpoint c that lies in I. Then

f is continuous at c

if and only if

$$\lim_{x \to c^+} f(x) = f(c).$$

Remarks

1. In Theorem F8, the interval I can have any of the forms $[c, \infty)$, $[c, b)$ or $[c, b]$, where $b > c$.
2. There is an analogous result to Theorem F8 for left limits.

It follows from Theorem F8 that $\lim\limits_{x \to 0^+} \sqrt{x} = 0$, as claimed earlier, since $f(x) = \sqrt{x}$ has domain $[0, \infty)$ and is continuous at 0.

Exercise F5

Prove the following.

(a) $\displaystyle \lim_{x \to 0^+} \left(\frac{\sin x}{x} + \sqrt{x} \right) = 1$ (b) $\displaystyle \lim_{x \to 0^+} \frac{\sin \sqrt{x}}{\sqrt{x}} = 1$

2 Asymptotic behaviour of functions

In our discussion of graph sketching in Unit A4 *Real functions, graphs and conics*, we described several types of *asymptotic behaviour* (that is, behaviour of a function when the domain variable or codomain variable becomes arbitrarily large), such as:

$$\frac{1}{x} \to \infty \text{ as } x \to 0^+ \quad \text{and} \quad e^x \to \infty \text{ as } x \to \infty.$$

In this section we define such statements formally and describe various relationships between them.

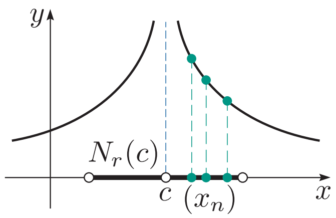


Figure 8 A function for which $f(x)$ tends to infinity as x tends to c

2.1 Functions which tend to infinity

In Section 1 we defined $f(x) \to l$ as $x \to c$ for a finite limit l in terms of the behaviour of sequences. We can define

$$f(x) \to \infty \text{ as } x \to c$$

in a similar way. This is illustrated in Figure 8.

Definition

Let the function f be defined on $N_r(c)$, for some $r > 0$. Then $f(x)$ **tends to ∞ as x tends to c** if

for each sequence (x_n) in $N_r(c)$ such that $x_n \to c$,
$f(x_n) \to \infty$.

In this case, we write

$$f(x) \to \infty \text{ as } x \to c.$$

Remarks

1. In this module we do not use the notation $\lim_{x \to c} f(x) = \infty$ as this can give the misleading impression that infinity can be treated in the same way as a finite limit. Algebraic manipulations of expressions involving ∞ are a common error in false proofs – as you saw, for example, for a series that is not convergent in Worked Exercise D29 in Unit D3 *Series*.
2. The statements
$$f(x) \to -\infty \text{ as } x \to c,$$
$$f(x) \to \infty \text{ (or } -\infty) \text{ as } x \to c^+ \text{ (or } c^-),$$
are defined similarly, with ∞ replaced by $-\infty$ and $N_r(c)$ replaced by the open interval $(c, c + r)$ or $(c - r, c)$, where $r > 0$, as appropriate.

There is a version of the Reciprocal Rule which relates functions that tend to infinity and functions that tend to 0. (You met the Reciprocal Rule for sequences in Subsection 4.3 of Unit D2.)

Theorem F9 Reciprocal Rule for limits

If the function f satisfies the conditions

1. $f(x) > 0$ for $x \in N_r(c)$, for some $r > 0$
2. $f(x) \to 0$ as $x \to c$,

then

$$1/f(x) \to \infty \quad \text{as } x \to c.$$

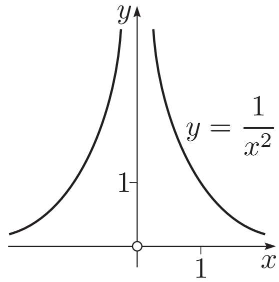


Figure 9 The graph of $y = 1/x^2$

For example,

$$1/x^2 \to \infty \quad \text{as } x \to 0,$$

because $f(x) = x^2 > 0$ for $x \in \mathbb{R} - \{0\}$, and $\lim\limits_{x \to 0} x^2 = 0$; see Figure 9.

The Reciprocal Rule can also be applied with $x \to c$ replaced by $x \to c^+$ or $x \to c^-$, and $N_r(c)$ replaced by $(c, c + r)$ or $(c - r, c)$, as appropriate. For example, we have

$$1/x \to \infty \quad \text{as } x \to 0^+,$$

because $f(x) = x > 0$ for $x \in (0, \infty)$, and $\lim\limits_{x \to 0^+} x = 0$; see Figure 10.

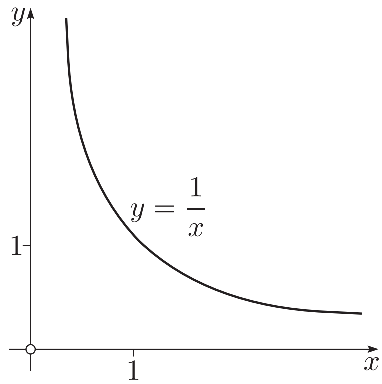


Figure 10 The graph of $y = 1/x$

Exercise F6

Prove that

(a) $\dfrac{1}{|x|} \to \infty$ as $x \to 0$ (b) $\dfrac{\sin x}{x^3} \to \infty$ as $x \to 0$

(c) $\dfrac{1}{x^3 - 1} \to \infty$ as $x \to 1^+$.

There are also versions of the Combination Rules and the Squeeze Rule for functions which tend to ∞ (or $-\infty$) as x tends to c, c^+ or c^-. Here we state the Combination Rules for functions which tend to ∞ as x tends to c.

Theorem F10 Combination Rules for functions which tend to infinity

If $f(x) \to \infty$ as $x \to c$ and $g(x) \to \infty$ as $x \to c$, then:

Sum Rule $f(x) + g(x) \to \infty$ as $x \to c$

Multiple Rule $\lambda f(x) \to \infty$ as $x \to c$, for $\lambda \in \mathbb{R}^+$

Product Rule $f(x)g(x) \to \infty$ as $x \to c$.

These rules are analogous to the corresponding rules for sequences which tend to infinity; see Subsection 4.3 of Unit D2.

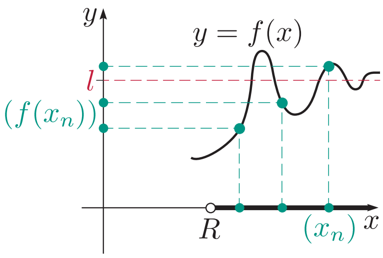


Figure 11 A function for which $f(x) \to l$ as $x \to \infty$

2.2 Behaviour as x tends to infinity

Next, we define various types of behaviour of real functions $f(x)$ as $x \to \infty$ or as $x \to -\infty$. To avoid repetition, in the following definition we allow the letter l to denote either a real number or one of the symbols ∞ or $-\infty$. The definition is illustrated in Figure 11.

Definition

Let the function f be defined on (R, ∞), for some real number R. Then $f(x)$ **tends to l as x tends to ∞** if

for each sequence (x_n) in (R, ∞) such that $x_n \to \infty$,
$f(x_n) \to l$.

In this case, we write

$$f(x) \to l \quad \text{as } x \to \infty.$$

The statement

$$f(x) \to l \quad \text{as } x \to -\infty$$

is defined similarly, with ∞ replaced by $-\infty$, and (R, ∞) replaced by $(-\infty, R)$. Note that

$$f(x) \to l \quad \text{as } x \to -\infty$$

is equivalent to

$$f(-x) \to l \quad \text{as } x \to \infty.$$

When l is a real number, we also use the notations

$$\lim_{x \to \infty} f(x) = l \quad \text{and} \quad \lim_{x \to -\infty} f(x) = l.$$

Once again, we can use versions of the Reciprocal Rule and the Combination Rules to obtain results about the behaviour of given functions as $x \to \infty$ or $-\infty$. The new versions of these rules are obtained from the Reciprocal Rule and the Combination Rules in Subsection 2.1 and the Combination Rules in Subsection 1.3 by replacing c by ∞ or $-\infty$, and $N_r(c)$ by (R, ∞) or $(-\infty, R)$, as appropriate.

Many results about the behaviour of functions $f(x)$ as x tends to ∞ or $-\infty$ are derived from the following two basic facts, often by using the Combination Rules and the Reciprocal Rule.

Theorem F11 Basic asymptotic behaviour

If $n \in \mathbb{N}$, then

(a) $x^n \to \infty$ as $x \to \infty$

(b) $\dfrac{1}{x^n} \to 0$ as $x \to \infty$.

We can use Theorem F11, together with the Combination Rules and the Reciprocal Rule, to determine the asymptotic behaviour of various functions defined by quotients. This is similar to determining the behaviour of sequences defined by quotients (see Subsection 3.2 of Unit D2), and we give a corresponding definition of the *dominant term* of a quotient that suits the present context.

Definition

The **dominant term** of a quotient involving the real variable x is the term in x (without its coefficient) which eventually has the largest absolute value.

For example, consider the behaviour of $x/(x^2+1)$ as $x \to \infty$. Here the dominant term is x^2, so we divide both the numerator and the denominator by x^2 to give

$$\frac{x}{x^2+1} = \frac{1/x}{1+1/x^2} \to \frac{0}{1+0} = 0 \quad \text{as } x \to \infty,$$

by Theorem F11(b) and the Combination Rules.

Exercise F7

Prove that:

(a) $\displaystyle\lim_{x\to\infty} \frac{2x^3+x}{x^3} = 2$ (b) $\dfrac{2x^3+1}{x^2} \to \infty$ as $x \to \infty$.

There are also versions of the Squeeze Rule for functions as x tends to infinity, which have some important applications. You met the corresponding versions of the Squeeze Rule for sequences in Theorems D10 and D18 in Sections 3 and 4 of Unit D2.

Theorem F12 Squeeze Rule for functions as $x \to \infty$

Let f, g and h be functions defined on some interval (R, ∞).

(a) If f, g and h satisfy the conditions

1. $g(x) \leq f(x) \leq h(x)$, for $x \in (R, \infty)$
2. $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = l$

where l is a real number, then

$$\lim_{x\to\infty} f(x) = l.$$

(b) If f and g satisfy the conditions

1. $f(x) \geq g(x)$, for $x \in (R, \infty)$
2. $g(x) \to \infty$ as $x \to \infty$

then

$$f(x) \to \infty \quad \text{as } x \to \infty.$$

Exercise F8

Use the Squeeze Rule to determine the behaviour of the following function as $x \to \infty$.

$$f(x) = \frac{\sin(1/x)}{x}$$

Hint: Use the fact that $-1 \leq \sin(1/x) \leq 1$ for $x \neq 0$.

In Subsection 1.3 we gave the Composition Rule for limits and Strategy F2 for using it. We can also use this strategy to deduce the asymptotic behaviour of composites of functions which have any of the types of asymptotic behaviour introduced in this unit, provided that we allow the letters l and L to denote either a real number or one of the symbols ∞ or $-\infty$. Notice that if l is either ∞ or $-\infty$, then the first proviso of the Composition Rule is automatically satisfied, so there is no need to check this.

For example, consider the asymptotic behaviour of $\dfrac{\sin(1/x^2)}{x^2}$ as $x \to \infty$. We can write

$$\frac{\sin(1/x^2)}{x^2} = g(f(x)),$$

where $f(x) = x^2$ and $g(x) = \dfrac{\sin(1/x)}{x}$. Substituting $u = f(x) = x^2$, we have

$$u = f(x) = x^2 \to \infty \quad \text{as } x \to \infty \quad \text{(by Theorem F11)},$$

$$g(u) = \frac{\sin(1/u)}{u} \to 0 \quad \text{as } u \to \infty \quad \text{(by Exercise F8)}.$$

Thus we deduce by Strategy F2 that

$$g(f(x)) = \frac{\sin(1/x^2)}{x^2} \to 0 \text{ as } x \to \infty.$$

In the next theorem, we collect together several standard results about the behaviour of particular functions as $x \to \infty$.

Theorem F13

(a) If $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$, where $n \in \mathbb{N}$, and

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

then

$$p(x) \to \infty \text{ as } x \to \infty \quad \text{and} \quad \frac{1}{p(x)} \to 0 \text{ as } x \to \infty.$$

(b) For each $n = 0, 1, 2, \ldots$, we have

$$\frac{e^x}{x^n} \to \infty \text{ as } x \to \infty \quad \text{and} \quad \frac{x^n}{e^x} \to 0 \text{ as } x \to \infty.$$

(c) We have

$$\log x \to \infty \text{ as } x \to \infty,$$

but, for each constant $a > 0$, we have

$$\frac{\log x}{x^a} \to 0 \text{ as } x \to \infty.$$

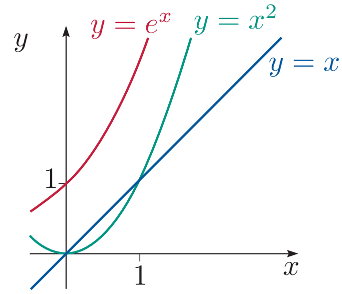


Figure 12 The graphs of $y = e^x$, $y = x^2$ and $y = x$

Part (b) tells us that, as x tends to infinity, e^x tends to infinity *faster* than any positive integer power of x, as illustrated in Figure 12. On the other hand, part (c) tells us that, as x tends to infinity, $\log x$ tends to infinity *more slowly* than any positive power of x. Thus, in part (b) the dominant term is e^x, whereas in part (c) it is x^a.

Proof of Theorem F13

(a) We use the fact that all the zeros of the polynomial p must lie in the interval $(-M, M)$, where $M = 1 + \max\{|a_{n-1}|, \ldots, |a_1|, |a_0|\}$, and

$$p(x) > 0, \quad \text{for } x \in (M, \infty). \tag{5}$$

This was proved in Theorem D54 of Unit D4.

Now for $x \neq 0$,

$$p(x) = x^n \left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right).$$

By Theorem F11(b) and the Combination Rules,

$$1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n} \to 1 + 0 + \cdots + 0 = 1 \text{ as } x \to \infty. \tag{6}$$

Thus, for $x \in (M, \infty)$, we have

$$\frac{1}{p(x)} = \frac{1/x^n}{1 + a_{n-1}/x + \cdots + a_0/x^n} \to \frac{0}{1} = 0 \quad \text{as } x \to \infty,$$

by statement (6), Theorem F11(b) and the Quotient Rule for limits. We deduce, by inequality (5) and the Reciprocal Rule, that

$$p(x) \to \infty \quad \text{as } x \to \infty.$$

(b) Let n be a fixed non-negative integer. We use the series representation

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \cdots,$$

which was shown to be valid for $x \geq 0$ in Subsection 4.1 of Unit D3. Since $x \geq 0$, all the terms in the above series are non-negative, so

$$e^x \geq \frac{x^{n+1}}{(n+1)!}, \qquad \text{for } x \geq 0.$$

Hence, for $x > 0$,

$$\frac{e^x}{x^n} \geq \frac{x}{(n+1)!} \quad \text{and} \quad 0 \leq \frac{x^n}{e^x} \leq \frac{(n+1)!}{x}.$$

It follows by Theorem F11, the Multiple Rule and the Squeeze Rule that

$$\frac{e^x}{x^n} \to \infty \quad \text{as } x \to \infty \quad \text{and} \quad \frac{x^n}{e^x} \to 0 \quad \text{as } x \to \infty.$$

(c) It was shown in Subsection 4.2 of Unit D4 that the function $x \longmapsto \log x$ is a strictly increasing inverse of the exponential function, with domain $(0, \infty)$ and range $\mathbb{R}$. We deduce that

$$\log x \to \infty \quad \text{as } x \to \infty.$$

Now let a be any positive constant. Since $x^a = \exp(a \log x)$, we make the substitution $t = a \log x$, so that $x^a = e^t$. For $x > 0$, this gives

$$\frac{\log x}{x^a} = \frac{t/a}{e^t} = \frac{t}{ae^t}. \tag{7}$$

Since $a > 0$, we have

$$t = a \log x \to \infty \quad \text{as } x \to \infty, \tag{8}$$

and, by part (b) with $n = 1$ and the Multiple Rule,

$$\frac{t}{ae^t} \to 0 \quad \text{as } t \to \infty. \tag{9}$$

Hence, by statements (7), (8) and (9), together with the Composition Rule, we have

$$\frac{\log x}{x^a} = \frac{t}{ae^t} \to 0 \quad \text{as } x \to \infty,$$

as required. ■

Exercise F9

Use the results of Theorem F13 and appropriate rules to determine the behaviour of the following functions as $x \to \infty$.

(a) $f(x) = \dfrac{e^x}{x^2} + \dfrac{3x^2}{\log x}$ (b) $f(x) = \dfrac{\log x}{e^x}$ (c) $f(x) = \dfrac{2e^x - x^2}{e^x + \log x}$

Hint: In part (b), express $(\log x)/e^x$ in terms of $(\log x)/x$ and x/e^x.

Exercise F10

Prove that:

(a) $e^{x^2}/x^2 \to \infty$ as $x \to \infty$ (b) $\log(\log x) \to \infty$ as $x \to \infty$

(c) $x \sin(1/x) \to 1$ as $x \to \infty$.

Hint: In part (c), use the substitution $u = 1/x$.

3 Continuity: the classical definition

In Book D you met a definition of continuity based on sequences, and saw that most familiar functions are continuous on their domains, a fact which is not at all surprising. However, there are many functions of interest for which it is more difficult to establish continuity (or discontinuity). Consider, for instance, the **Riemann function**, which has domain $\mathbb{R}$ and rule

$$f(x) = \begin{cases} 1/q, & \text{if } x \text{ is a rational } p/q, \text{ where } q > 0, \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Note that, in this section, we assume that all rationals p/q are expressed in lowest terms; that is, the greatest common factor of p and q is 1.

It follows from the definition that, for example, $f(2/3) = 1/3$ and $f(\sqrt{2}) = 0$. By plotting the values of $f(x)$ at different values of x we can produce a sketch of the graph of the Riemann function as shown in Figure 13.

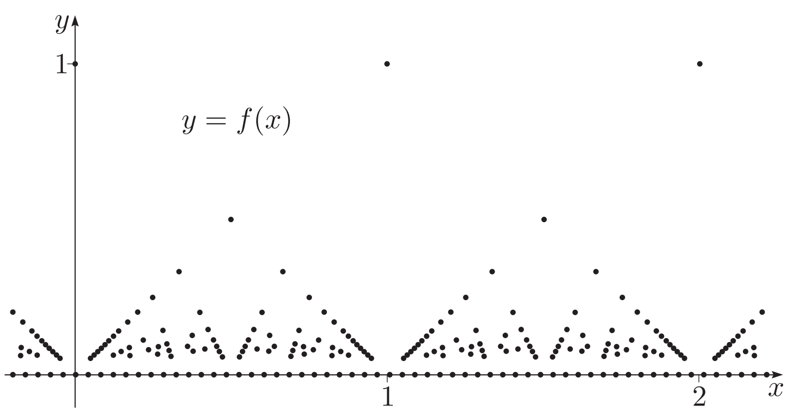


Figure 13 A sketch of the graph of the Riemann function

From this sketch of the graph of f it is not clear whether the Riemann function is continuous at any point of $\mathbb{R}$, but you will see (in Subsection 3.2) that in fact it is continuous at infinitely many points and discontinuous at infinitely many points! When dealing with such unusual functions, it is useful to have available the alternative definition of continuity which is introduced in Subsection 3.1. This definition looks more abstract but is more effective in some cases.

The emergence of rigorous analysis

The classical definition of continuity described in this section emerged towards the end of the nineteenth century after many years of debate amongst mathematicians about the rigorous formulation of analysis as the foundation of calculus. At this time, the informal approach used in the eighteenth century, for instance by Euler, was increasingly found to be inadequate. For example, around 1820, Joseph Fourier (1768–1830) used functions defined by infinite series to solve problems in the theory of heat. The properties of these series raised challenging questions about the meaning of convergence. This led to questions about the definitions of continuity, limits, differentiation and integration, and even the nature of the real numbers. These questions were not properly resolved until about 1870, after contributions by many mathematicians, including Bolzano, Cauchy, Riemann, Dirichlet, Dedekind, Weierstrass and Cantor.

3.1 The ε-δ definition of continuity

The sequential definition of continuity that you studied in Unit D4 states that the function $f : A \longrightarrow \mathbb{R}$ is continuous at c, where $c \in A$, if

for each sequence (x_n) in A such that $x_n \to c$,
$f(x_n) \to f(c)$.

This definition uses sequences to formalise the intuitive idea that $f(x)$ approaches $f(c)$ as x approaches the point c in any manner.

The new definition that we study in this unit formalises this idea in a somewhat different way, which we can describe in words as follows:

we can make $f(x)$ as close as we wish to $f(c)$ by ensuring that x is close enough to c.

The 'closeness' in this description is measured by two variables, ε in the codomain and δ in the domain, which represent 'small' positive numbers.

Definition

Let the function f have domain A and let $c \in A$. Then f is **continuous** at c if

$$\text{for each } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that}$$
$$|f(x) - f(c)| < \varepsilon, \quad \text{for all } x \in A \text{ with } |x - c| < \delta. \tag{10}$$

Remarks

1. The above definition is quite subtle. It can be expressed in words as follows: no matter how small a positive number ε we are *given*, we can *choose* a positive number δ such that if the distance between x and c is less than δ, then the distance between $f(x)$ and $f(c)$ is less than ε. Thus statement (10) can be interpreted as an *implication*:

   $$\text{if } x \in A \text{ and } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \varepsilon.$$

2. Note that

   $$\begin{array}{lll} |f(x) - f(c)| < \varepsilon & \text{is equivalent to} & f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \\ |x - c| < \delta & \text{is equivalent to} & c - \delta < x < c + \delta, \end{array}$$

   as illustrated in Figure 14. This shows that as x gets closer to c, $f(x)$ gets closer to $f(c)$.

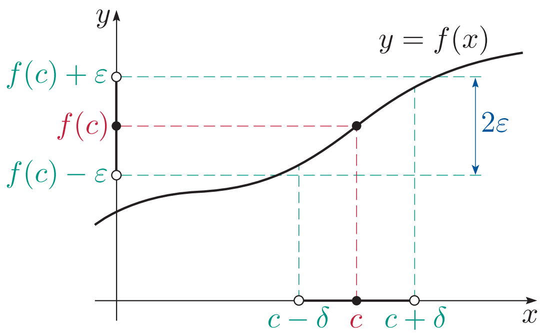


Figure 14 The ε-δ definition of continuity

3. Usually the value of δ that we choose in order to make statement (10) true depends on the given value of ε: the smaller ε is, the smaller δ has to be. The value of δ often depends also on the particular point c at which we are checking continuity.

We can interpret the task of finding a suitable choice of δ, when using this definition, as an 'ε-δ game' in which player A chooses a small positive number ε and then challenges player B to find a suitably small positive number δ for which statement (10) is true. (This is like the 'ε-N game' for null sequences that you met in Unit D2.) For example, suppose that $f(x) = x^2$ and $c = 0$. If player A chooses $\varepsilon = 1/4$ then player B can choose $\delta = 1/2$ (or any smaller value) because if

$$|x - 0| = |x| < 1/2,$$

then

$$|f(x) - f(0)| = |x^2| < 1/4.$$

This is illustrated in Figure 15.

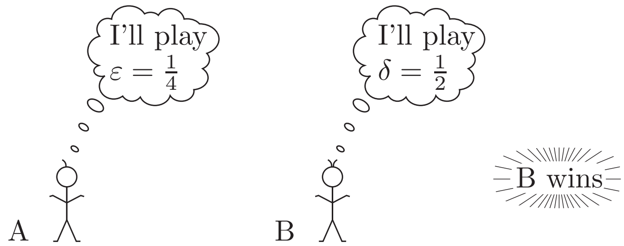


Figure 15 The ε-δ game for $f(x) = x^2$ at $c = 0$

In practice, we do not usually choose specific values for ε when proving continuity, but this game illustrates the ideas involved.

The method of applying the ε-δ definition of continuity depends on the nature of the function f. As an illustration, we apply the definition to polynomial functions, using the following strategy.

Strategy F3

To use the ε-δ definition to prove that a polynomial function f with domain A is continuous at a point $c \in A$, let $\varepsilon > 0$ be given and carry out the following steps.

1. Use algebraic manipulation to express the difference $f(x) - f(c)$ as a product of the form $(x - c)g(x)$.
2. Obtain an upper bound of the form $|g(x)| \leq M$, for $|x - c| \leq r$, where $r > 0$ is chosen so that $[c - r, c + r] \subset A$. (The Triangle Inequality is often useful here.)
3. Use the fact that $|f(x) - f(c)| \leq M|x - c|$, for $|x - c| \leq r$, to choose $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon, \quad \text{for all } x \in A \text{ with } |x - c| < \delta.$$

The following worked exercise shows how Strategy F3 can be used.

Worked Exercise F5

Use the ε-δ definition to prove that $f(x) = x^2$ is continuous at $c = 2$.

Solution

The domain of f is $\mathbb{R}$.

Let $\varepsilon > 0$ be given. We want to choose $\delta > 0$, in terms of ε, such that

$$|f(x) - f(2)| < \varepsilon, \quad \text{for all } x \text{ with } |x-2| < \delta. \qquad (*)$$

We follow the steps in Strategy F3.

1. First we write

   $$f(x) - f(2) = x^2 - 4 = (x-2)(x+2).$$

   Writing f in this form will help us to show that if $|x-2|$ is small, then $|f(x) - f(2)|$ is small.

2. Next we obtain an upper bound for $|x+2|$ when x is near 2.

   We consider points for which $|x-2| \leq r$ with $r = 1$; any $r > 0$ is suitable, but the resulting bounds will depend on r.

   If $|x-2| \leq 1$, then x lies in the closed interval $[1, 3]$, so

   $$\begin{aligned} |x+2| &\leq |x| + 2 \qquad \text{(by the Triangle Inequality)} \\ &\leq 3 + 2 = 5. \end{aligned}$$

3. Hence

   $$|f(x) - f(2)| \leq 5|x-2|, \quad \text{for } |x-2| \leq 1.$$

   So if $|x-2| < \delta$, where $0 < \delta \leq 1$, then

   $$|f(x) - f(2)| < 5\delta.$$

   Now $5\delta \leq \varepsilon$ if and only if $\delta \leq \frac{1}{5}\varepsilon$.

   We now choose δ so that *both* of the inequalities $0 < \delta \leq 1$ and $0 < \delta \leq \frac{1}{5}\varepsilon$ are satisfied.

   Thus, if we choose $\delta = \min\left\{1, \frac{1}{5}\varepsilon\right\}$, then

   $$|f(x) - f(2)| < 5\delta \leq 5 \times \tfrac{1}{5}\varepsilon = \varepsilon, \quad \text{for all } x \text{ with } |x-2| < \delta,$$

   which proves statement $(*)$.

Thus f is continuous at the point 2.

Exercise F11

Use the ε-δ definition to prove that $f(x) = x^3$ is continuous at $c = 1$.

Hint: Note that $x^3 - 1 = (x-1)(x^2 + x + 1)$.

Next we verify that the two definitions of continuity are equivalent.

Theorem F14

The ε-δ definition and the sequential definition of continuity are equivalent.

Proof Let the function f have domain A, with $c \in A$. First we assume that f is continuous at c according to the ε-δ definition. We want to deduce that,

for each sequence (x_n) in A such that $x_n \to c$,

$$f(x_n) \to f(c). \tag{11}$$

Let $\varepsilon > 0$ be given. Then, by the ε-δ definition of continuity, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon, \quad \text{for all } x \in A \text{ with } |x - c| < \delta. \tag{12}$$

Since $x_n \to c$, there exists an integer N such that

$$|x_n - c| < \delta, \quad \text{for all } n > N.$$

Hence, by statement (12),

$$|f(x_n) - f(c)| < \varepsilon, \quad \text{for all } n > N.$$

Thus statement (11) does indeed hold, so the sequential definition follows from the ε-δ definition.

Next suppose that f is continuous at c according to the sequential definition. We want to deduce that if $\varepsilon > 0$ is given, then there exists $\delta > 0$ such that statement (12) holds.

We use a proof by contradiction.

Suppose that, for some $\varepsilon > 0$, there is *no* such $\delta > 0$. Then statement (12) must be false with $\delta = 1$, $\delta = \frac{1}{2}$, $\delta = \frac{1}{3}$, and so on. Hence, for each $n \in \mathbb{N}$, there exists $x_n \in A$ with $|x_n - c| < 1/n$ such that

$$|f(x_n) - f(c)| \geq \varepsilon. \tag{13}$$

Now, the sequence (x_n) lies in A and $x_n \to c$. Thus, by the sequential definition of continuity, we have $\lim_{n\to\infty} f(x_n) = f(c)$, which contradicts inequality (13). We deduce that the ε-δ definition of continuity follows from the sequential definition. ■

3.2 Continuity of some unusual functions

It is natural to ask which is the 'better' definition of continuity. It is difficult to give a definitive answer, but on the whole:

- when proving the continuity of simpler functions the sequential definition is usually easier, whereas the ε-δ definition can work better with more complicated functions
- when proving discontinuity the sequential definition is usually easier.

For most of the functions you have met so far in this module, the points where the functions are continuous have been 'obvious'. However, there are many functions for which it is far less clear where they are continuous, if anywhere. In this subsection you will meet several interesting but quite complicated functions, and you will see that the ε-δ definition is an effective means of proving continuity, even when it is not possible to use Strategy F3.

The proofs in this subsection will take some effort to understand fully, so you may prefer to skim through them on a first reading and return to them as time permits. Do not be discouraged if you find them rather hard at first: reading proofs in analysis gets easier as you become more familiar with the sorts of arguments used.

The Dirichlet function and the Riemann function

The first function we consider has a simple definition, but is highly discontinuous. The **Dirichlet function** has domain $\mathbb{R}$ and rule

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

The graph of f in Figure 16 looks rather like two parallel lines, but each line has infinitely many 'holes' in it!

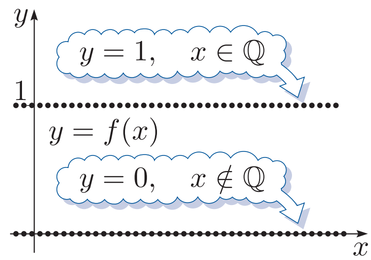


Figure 16 The Dirichlet function

Theorem F15

The Dirichlet function is discontinuous at every point of $\mathbb{R}$.

Proof Let c be any point of $\mathbb{R}$. We show that f is discontinuous at c by using the sequential definition of continuity.

We do this by showing that there are sequences that tend to c whose images under f tend to different limits. Recall the Density Property of $\mathbb{R}$ from Subsection 1.4 of Unit D1 *Numbers*: between any two real numbers, we can find both a rational number and an irrational number.

By the Density Property of $\mathbb{R}$, each open interval of the form

$$(c - 1/n, c + 1/n), \quad \text{where } n \in \mathbb{N},$$

contains a rational x_n and an irrational y_n. Considering the sequences (x_n) and (y_n), we have $x_n \to c$ and $y_n \to c$ by the Squeeze Rule for sequences, but

$$f(x_n) = 1 \quad \text{and} \quad f(y_n) = 0, \quad \text{for } n = 1, 2, \ldots.$$

Since $(f(x_n))$ and $(f(y_n))$ have different limits, f is discontinuous at c. ■

Our next function shows even stranger behaviour. The Riemann function, which we introduced at the start of this section, has domain $\mathbb{R}$ and rule

$$f(x) = \begin{cases} 1/q, & \text{if } x \text{ is a rational } p/q, \text{ where } q > 0, \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

(Recall that, in this section, p/q is always expressed in its lowest terms.)

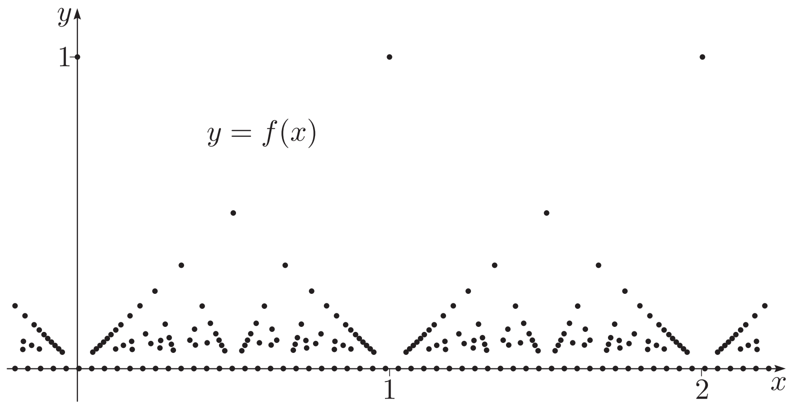


Figure 17 A sketch of the Riemann function

As mentioned earlier, it is not clear from the sketch of the graph of f in Figure 17 whether the Riemann function is continuous at any point of $\mathbb{R}$. In fact, it has the remarkable property that each open interval of $\mathbb{R}$ contains infinitely many points where f is continuous and infinitely many points where f is discontinuous. As you study the proof, notice how it illustrates the strengths of the two definitions of continuity.

Theorem F16

The Riemann function is discontinuous at each rational point of $\mathbb{R}$ and continuous at each irrational point.

Proof Here we prove discontinuity using the sequential definition and we prove continuity using the ε-δ definition.

First we prove that f is discontinuous at rational points.

Here we use Strategy D14 from Unit D4; that is, for each rational point c, we find a sequence that converges to c but whose images under f do not converge to $f(c)$.

Let $c = p/q$, with $q > 0$ (where p/q is expressed in lowest terms). Then, by the Density Property of $\mathbb{R}$, each open interval of the form

$$(c - 1/n, c + 1/n), \quad \text{where } n \in \mathbb{N},$$

contains an irrational number x_n. Considering the sequence (x_n), we have $x_n \to c$ and $f(x_n) = 0$, for $n = 1, 2, \ldots$. Since $f(c) = 1/q \neq 0$, we have $f(x_n) \nrightarrow f(c)$, so f is discontinuous at c.

Recall that the notation $\nrightarrow$ is read as 'does not tend to'.

Next we prove that f is continuous at irrational points. Let c be an irrational number in $\mathbb{R}$. We must prove that

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon, \quad \text{for all } x \text{ with } |x - c| < \delta. \tag{14}$$

Let $\varepsilon > 0$ be given. Since c is irrational, we have $f(c) = 0$. Also, $f(x) \geq 0$ for all x in $\mathbb{R}$, so statement (14) can be rewritten as

$$f(x) < \varepsilon, \quad \text{for all } x \text{ with } |x - c| < \delta. \tag{15}$$

Note that Strategy F3 cannot be used here, since f is not a polynomial function.

To obtain a value of δ such that statement (15) holds, we first choose a positive integer N such that $1/N < \varepsilon$. Then we let S_N denote the set of rationals p/q in the interval $(c-1, c+1)$ such that $0 < q \leq N$. There are only finitely many elements of the set S_N, and $c \notin S_N$ because c is irrational. Thus the number

$$\delta = \min\{|x - c| : x \in S_N\}$$

exists and is positive. Therefore the open interval $(c - \delta, c + \delta)$ contains *no* rationals p/q with $0 < q \leq N$.

Hence if $|x - c| < \delta$, then

either $\quad x$ is irrational, so $f(x) = 0 < \varepsilon$,

or $\quad x = p/q$ with $q > N$, so $f(x) = 1/q < 1/N < \varepsilon$.

In either case $f(x) < \varepsilon$, so we have succeeded in choosing $\delta > 0$ such that statement (15) holds. Hence f is continuous at c. ■

In view of the strange properties of the Riemann function, it is natural to ask whether a function can also be found which is continuous at each rational point of $\mathbb{R}$ and discontinuous at each irrational point. It can be shown that no such function exists, but we do not prove this here.

The blancmange function

Our next function is in some ways even more unusual than the Dirichlet and the Riemann functions. To construct this function, we start with the **sawtooth function** illustrated in Figure 18:

$$s(x) = \begin{cases} x - \lfloor x \rfloor, & \text{if } 0 \leq x - \lfloor x \rfloor \leq \frac{1}{2}, \\ 1 - (x - \lfloor x \rfloor), & \text{if } \frac{1}{2} < x - \lfloor x \rfloor < 1, \end{cases}$$

where $\lfloor x \rfloor$ is the integer part function.

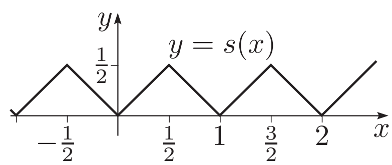


Figure 18 The sawtooth function

The **blancmange function** B is obtained by forming an infinite series of functions related to s:

$$\begin{aligned} B(x) &= s(x) + \tfrac{1}{2}s(2x) + \tfrac{1}{4}s(4x) + \tfrac{1}{8}s(8x) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n} s(2^n x). \end{aligned}$$

Teiji Takagi

The properties of this function were first studied by the Japanese mathematician Teiji Takagi (1875–1960) in 1903. The name 'blancmange function' was used by the English mathematician David Tall (1941–) in the 1980s.

For example, to evaluate $B\left(\frac{1}{4}\right)$ we find the sum of the corresponding series:

$$\begin{aligned} B\left(\tfrac{1}{4}\right) &= s\left(\tfrac{1}{4}\right) + \tfrac{1}{2}s\left(\tfrac{1}{2}\right) + \tfrac{1}{4}s(1) + \tfrac{1}{8}s(2) + \cdots \\ &= \tfrac{1}{4} + \tfrac{1}{2} \times \tfrac{1}{2} + \tfrac{1}{4} \times 0 + \tfrac{1}{8} \times 0 + \cdots \\ &= \tfrac{1}{2}. \end{aligned}$$

In this case, the series has only finitely many non-zero terms, but for some x the series for $B(x)$ has infinitely many non-zero terms. However, since $0 \leq s(x) \leq \frac{1}{2}$, for $x \in \mathbb{R}$, we have

$$0 \leq \frac{1}{2^n} s(2^n x) \leq \frac{1}{2^{n+1}}, \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{N},$$

so the series defining $B(x)$ is convergent for each $x \in \mathbb{R}$, by the Comparison Test for series; see Subsection 2.1 of Unit D3.

To picture the graph of the blancmange function, we consider the graphs of several successive partial sum functions of the series for B, with domains restricted to $[0, 1]$; see Figure 19. (In each case the graph of the previous partial sum function is in light dashes and the function being added is in heavy dashes. Thus the solid line is the sum of the graphs shown by dashed lines.)

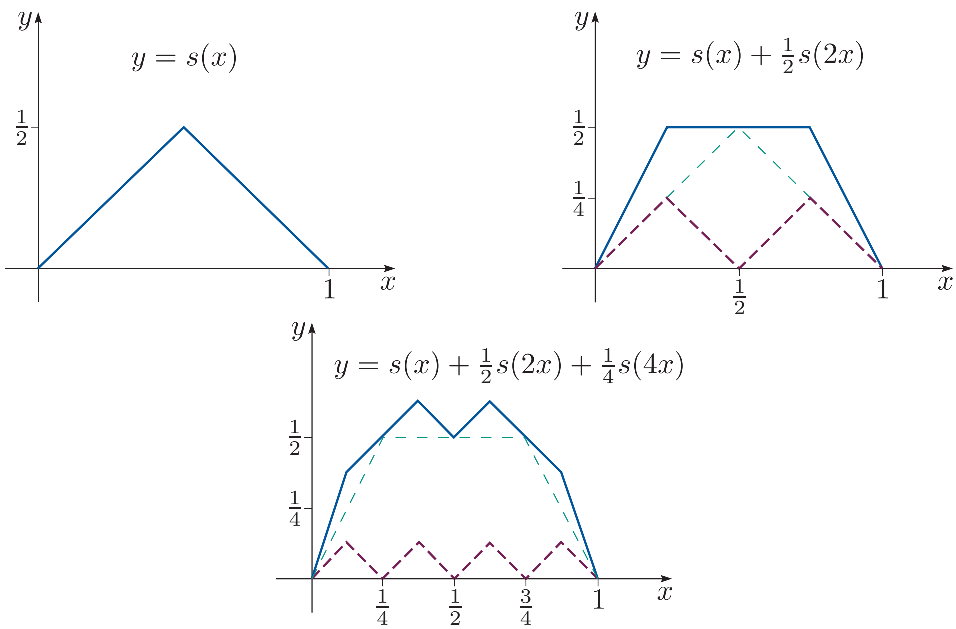


Figure 19 Successive steps in the construction of the blancmange function

The sum function B has the graph shown in Figure 20.

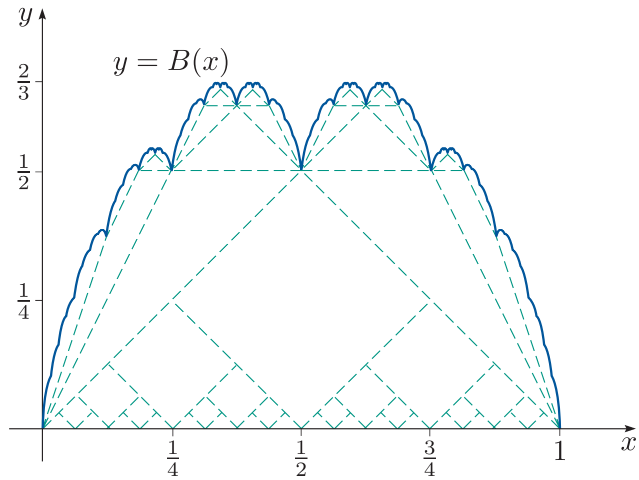


Figure 20 The blancmange function

The graph of B is very irregular, in the sense that it oscillates rapidly up and down, and does not appear to be smooth at any point. In fact, in Unit F2 you will see that the function B is nowhere differentiable! However, it does seem that the function B is continuous, and we can show that this is true.

Theorem F17

The blancmange function is continuous.

Proof We use the ε-δ definition of continuity.

Let $c \in \mathbb{R}$. We want to show that

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|B(x) - B(c)| < \varepsilon, \quad \text{for all } x \text{ with } |x - c| < \delta. \tag{16}$$

Let $\varepsilon > 0$ be given. We first write

$$B(x) - B(c) = \sum_{n=0}^{\infty} \frac{1}{2^n} \left(s(2^n x) - s(2^n c) \right).$$

Hence, by the infinite form of the Triangle Inequality (see Theorem D35 in Subsection 3.1 of Unit D3),

$$|B(x) - B(c)| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} \left| s(2^n x) - s(2^n c) \right|. \tag{17}$$

For all x and c, and $n = 0, 1, 2, \ldots$, both $s(2^n x)$ and $s(2^n c)$ lie in the interval $\left[0, \frac{1}{2}\right]$, so

$$|s(2^n x) - s(2^n c)| \leq \tfrac{1}{2}, \quad \text{for } n = 0, 1, 2, \ldots. \tag{18}$$

Now we choose an integer N such that $1/2^N < \frac{1}{2}\varepsilon$ and consider the 'tail' of the series in inequality (17), starting from the term $n = N$.

Such an N exists because $(1/2^n)$ is a basic null sequence. Splitting the series into two different parts will enable us to use different methods for each part. We bound each part by $\frac{1}{2}\varepsilon$ to give an overall bound of ε.

By inequality (18), we have

$$\begin{aligned}\sum_{n=N}^{\infty} \frac{1}{2^n}|s(2^n x) - s(2^n c)| &\leq \frac{1}{2}\sum_{n=N}^{\infty}\frac{1}{2^n}\\ &= \frac{1}{2}\left(\frac{1/2^N}{1-1/2}\right) = \frac{1}{2^N} < \tfrac{1}{2}\varepsilon. \qquad (19)\end{aligned}$$

Here we have used the fact that $\displaystyle\sum_{n=N}^{\infty}\frac{1}{2^n}$ is a geometric series with $a = 1/2^N$ and $r = 1/2$, so $\displaystyle\sum_{n=N}^{\infty}\frac{1}{2^n} = \frac{1/2^N}{1-1/2}$.

Next we consider the rest of this series. Each of the functions

$$x \longmapsto s(2^n x), \quad n = 0, 1, \dots, N-1,$$

is continuous.

We omit the proof of this – you may like to check it for yourself.

Therefore, for each $n = 0, 1, \dots, N-1$, there is a positive number δ_n such that

$$|s(2^n x) - s(2^n c)| < \tfrac{1}{4}\varepsilon, \quad \text{for all } x \text{ with } |x - c| < \delta_n.$$

You will see a bit later in the argument why it makes sense to choose $\frac{1}{4}\varepsilon$ here.

Thus if $\delta = \min\{\delta_0, \delta_1, \dots, \delta_{N-1}\}$ and $|x - c| < \delta$, then

$$\sum_{n=0}^{N-1}\frac{1}{2^n}|s(2^n x) - s(2^n c)| < \sum_{n=0}^{N-1}\frac{1}{2^n}\left(\tfrac{1}{4}\varepsilon\right) < 2 \times \tfrac{1}{4}\varepsilon = \tfrac{1}{2}\varepsilon.$$

Here we have used the fact that $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n} = 2$.

Combining this inequality with inequalities (19) and (17), we obtain statement (16) with this choice of δ. Hence B is continuous at the point c. ■

The blancmange function is very irregular, but it exhibits patterns known as 'self-similarity'. However closely you look at the graph, you can see 'mini-blancmanges' growing on it everywhere. The existence of these mini-blancmanges can be explained by rewriting the series defining B:

$$\begin{aligned}B(x) &= s(x) + \tfrac{1}{2}s(2x) + \tfrac{1}{4}s(4x) + \tfrac{1}{8}s(8x) + \cdots\\ &= s(x) + \tfrac{1}{2}\left(s(2x) + \tfrac{1}{2}s(4x) + \tfrac{1}{4}s(8x) + \cdots\right)\\ &= s(x) + \tfrac{1}{2}B(2x).\end{aligned}$$

The graph of the function $x \longmapsto \frac{1}{2}B(2x)$ is just the graph of B scaled by the factor $\frac{1}{2}$ in both x- and y-directions. Hence the graph of B is the graph of s with a (sheared) $\frac{1}{2}$-size blancmange growing on each sloping line segment. (You met shears in Unit C3 *Linear Transformations*.) Smaller mini-blancmanges can be explained in a similar manner. This is illustrated in Figure 21.

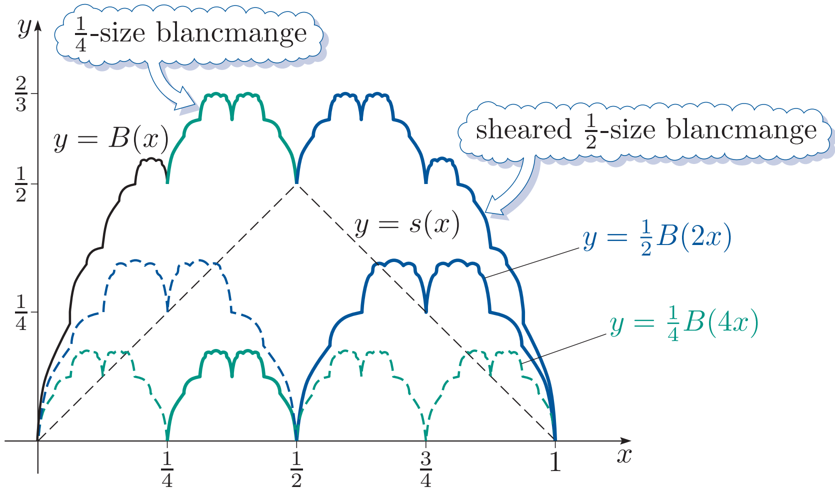


Figure 21 The self-similarity of the blancmange function

Such irregular sets, which display self-similarity, are studied in detail in the subject known as 'fractals'.

3.3 Limits and other asymptotic behaviour

In Sections 1 and 2 we defined limits and other types of asymptotic behaviour using a sequential approach. Each of these concepts can also be defined in a way that is analogous to the ε-δ definition of continuity. For example, we can define the concept of a limit as follows.

Definition

Let f be a function defined on a punctured neighbourhood $N_r(c)$ of c. Then $f(x)$ **tends to the limit l as x tends to c** if

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon, \quad \text{for all } x \text{ with } 0 < |x - c| < \delta.$$

As before, we write

$$\lim_{x \to c} f(x) = l \quad \text{or} \quad f(x) \to l \text{ as } x \to c.$$

Remarks

1. This definition is very similar to the ε-δ definition of continuity, except that $f(c)$ is replaced by l and $|x-c|<\delta$ is replaced by $0<|x-c|<\delta$. This reflects the fact that when we try to find the limit of $f(x)$ as x tends to c, the value of $f(x)$ at $x=c$ is not relevant and indeed may not be defined.
2. The proof that the above definition is equivalent to the sequential definition of a limit is similar to the proof of Theorem F14.
3. Analogous definitions can be constructed for one-sided limits and for all the types of asymptotic behaviour you have met.

   For example, $\lim\limits_{x\to\infty} f(x) = l$ if

   $$\text{for each } \varepsilon > 0, \text{ there exists } K \in \mathbb{R} \text{ such that}$$
   $$|f(x) - l| < \varepsilon, \quad \text{for all } x \text{ with } x > K.$$

A brief history of the limit concept

Recent historical studies have shown that, contrary to what was previously believed, Isaac Newton (1642–1727) had a good grasp of the limit concept, recognising that limits provide a secure foundation for calculus. Jean le Rond d'Alembert (1717–1783) in his article in the French *Encyclopédie* (published between 1751 and 1765) provided a definition of limit very close to that of Newton: a bound that could be approached as closely as one chose. However, because d'Alembert, like Newton, worked with examples that were primarily geometric, there was no need to consider quantities that might oscillate from one side of a limit to the other.

In 1821 Augustin-Louis Cauchy (1789–1857), in his *Cours d'Analyse*, provided a definition of a limit which combined the same ideas as that of d'Alembert: the existence of a fixed value and the possibility of approaching it as closely as one wishes. Although Cauchy was the first to use ε-δ arguments in his proofs, he never gave an explicit ε-δ definition of a limit. Unknown to Cauchy, in 1817 the Bohemian theologian and mathematician Bernard Bolzano (1781–1848) had already introduced a rigorous ε-δ definition of a limit, but his work was not well known and had only indirect influence on later developments. The ε-δ definition used today was first formalised by the German mathematician Karl Weierstrass (1815–1897) in his lectures in Berlin in the 1860s.

Karl Weierstrass

The next worked exercise gives an example of a limit evaluated using the ε-δ definition. (In fact, for this example, the sequential definition is easier to use; see Worked Exercise F1(a).)

Worked Exercise F6

Use the ε-δ definition of a limit to evaluate

$$\lim_{x\to 2} \frac{x^2-4}{x-2}.$$

Solution

The domain of $f(x) = (x^2-4)/(x-2)$ is $\mathbb{R} - \{2\}$, so f is defined on each punctured neighbourhood of 2. Also,

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2, \quad \text{for } x \neq 2.$$

Limits of this type with a factor that can be cancelled often arise in calculations.

This formula suggests that $\lim_{x\to 2} f(x) = 2+2 = 4$, so we must prove that

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - 4| < \varepsilon, \quad \text{for all } x \text{ with } 0 < |x-2| < \delta.$$

But $|f(x) - 4| = |x+2-4| = |x-2|$, for $x \neq 2$, so the above statement is true if we choose $\delta = \varepsilon$. Hence

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$$

This is similar to a proof, using Strategy F3, that the function $x \longmapsto x+2$ is continuous at the point 2.

Exercise F12

Use the ε-δ definition of a limit to evaluate

$$\lim_{x\to 1} \frac{2x^3+3x-5}{x-1}.$$

Hint: Use the fact that $2x^3+3x-5 = (x-1)(2x^2+2x+5)$ and follow Strategy F3 for using the ε-δ definition of continuity.

4 Uniform continuity

In Section 3 you met an alternative approach to continuity, based on the ε-δ definition. In this section you will see how this approach can be used to describe a stronger notion of continuity, which plays a key role in our later work on the integration of continuous functions in Unit F3 *Integration*.

4.1 What is uniform continuity?

The ε-δ definition of continuity states that a function f is continuous at a point c in an interval I in the domain of f if

$$\text{for each } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that}$$
$$|f(x) - f(c)| < \varepsilon, \quad \text{for all } x \in I \text{ with } |x - c| < \delta.$$

In this definition we cannot expect that, for a given positive number ε, the same positive number δ will serve equally well for each point c in I. Sometimes, however, this does happen, in which case:

$$\text{for each } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that}$$
$$|f(x) - f(c)| < \varepsilon, \quad \text{for all } x, c \in I \text{ with } |x - c| < \delta.$$

Informally, for all c in the interval I, as x gets closer to c, the values $f(x)$ get closer to $f(c)$ at least as quickly as some uniform rate. We make the following definition.

Definition

A function f defined on an interval I is **uniformly continuous** on I if

$$\text{for each } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that}$$
$$|f(x) - f(y)| < \varepsilon, \quad \text{for all } x, y \in I \text{ with } |x - y| < \delta. \tag{20}$$

Remarks

1. In this definition we have used the variables x and y, rather than x and c, to indicate that these two variables are of equal standing. Notice that uniform continuity is defined on an *interval*: it is meaningless to say that a function is uniformly continuous at a *point*.
2. We say that c is an **interior point** of an interval I if c is not an endpoint of I. It follows from the above definition that if f is uniformly continuous on an interval I, then f is continuous at each interior point of I. At an endpoint of I, the function can be discontinuous because of its behaviour *outside* I.
3. If f is uniformly continuous on an interval I, then f is uniformly continuous on any subinterval of I.

Worked Exercise F7

Prove from the definition that $f(x) = x^2$ is uniformly continuous on $I = [-4, 4]$.

Solution

Let $\varepsilon > 0$ be given. We have

$$f(x) - f(y) = x^2 - y^2 = (x + y)(x - y).$$

Hence, for $x, y \in [-4, 4]$,

$$\begin{aligned} |f(x) - f(y)| &= |x + y|\,|x - y| \\ &\leq (|x| + |y|)|x - y| \quad \text{(by the Triangle Inequality)} \\ &\leq 8|x - y|, \end{aligned}$$

since $|x| \leq 4$ and $|y| \leq 4$.

Thus, if we choose $\delta = \frac{1}{8}\varepsilon$, then whenever $x, y \in [-4, 4]$ and $|x - y| < \delta$, we have

$$\begin{aligned} |f(x) - f(y)| &\leq 8|x - y| \\ &< 8 \times \tfrac{1}{8}\varepsilon = \varepsilon. \end{aligned}$$

Hence f is uniformly continuous on $[-4, 4]$.

In Exercise F13(b) you will see that $f(x) = x^2$ is *not* uniformly continuous on $\mathbb{R}$. It is, however, uniformly continuous on all bounded closed intervals, as you will see in Subsection 4.2.

In the next worked exercise we consider the function $f(x) = 1/x$. In Unit D4 you saw that this function is continuous on its domain, and hence on $(0, 1]$. We now show that it is not *uniformly* continuous on $(0, 1]$.

Worked Exercise F8

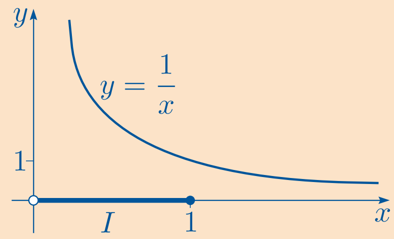
Prove that $f(x) = 1/x$ is not uniformly continuous on $I = (0, 1]$.

Solution

To prove that a function is *not* uniformly continuous, we need to show that the negation of statement (20) holds.

We have to find $\varepsilon > 0$ such that, no matter which $\delta > 0$ is chosen, there are points x and y in I with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

We sketch the graph of the function f.



The graph suggests that, for any positive δ, we should take x and y near 0 because then x and y are close together but $f(x)$ and $f(y)$ can be far apart.

We try $x = \frac{1}{2}\delta$ and $y = \delta$, where $0 < \delta < 1$. Then

$$|x - y| = \left|\tfrac{1}{2}\delta - \delta\right| = \tfrac{1}{2}\delta < \delta$$

and

$$|f(x) - f(y)| = \left|\frac{1}{\frac{1}{2}\delta} - \frac{1}{\delta}\right| = \frac{2}{\delta} - \frac{1}{\delta} = \frac{1}{\delta} > 1.$$

Hence the negation of statement (20) holds with $\varepsilon = 1$, so f is not uniformly continuous on I.

The reasoning in this solution is quite subtle and can be tricky to apply, so it is useful to reformulate what it means to say that a function is *not* uniformly continuous on an interval. Roughly speaking, this happens if you can find pairs of points in the interval, as close together as you like, whose images are *not* close together. We prove the following result, illustrated in Figure 22.

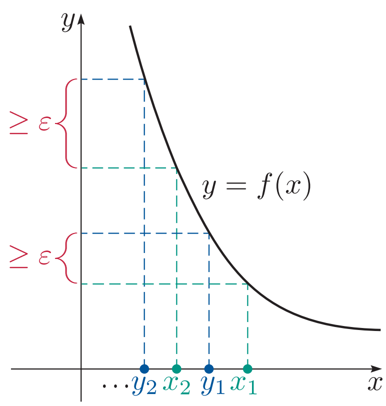


Figure 22 A function which is not uniformly continuous

Theorem F18

Let the function f be defined on an interval I. Then f is not uniformly continuous on I if and only if there exist two sequences (x_n) and (y_n) in I, and $\varepsilon > 0$, such that

1. $|x_n - y_n| \to 0$ as $n \to \infty$
2. $|f(x_n) - f(y_n)| \geq \varepsilon$, for $n = 1, 2, \ldots$.

Proof We start by proving the *only if* statement.

First suppose that f is not uniformly continuous on I. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there are points x and y in I with

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| \geq \varepsilon.$$

Applying this fact with $\delta = 1$, $\delta = \frac{1}{2}$, $\delta = \frac{1}{3}$, and so on, we obtain sequences (x_n) and (y_n) in I such that

$$|x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon, \quad \text{for } n = 1, 2, \ldots.$$

Thus statements 1 and 2 both hold.

On the other hand, suppose that there exist sequences (x_n) and (y_n) in I, and $\varepsilon > 0$, such that statements 1 and 2 hold.

We use a proof by contradiction to show that f is not uniformly continuous on I.

If f *is* uniformly continuous on I, then there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \quad \text{for all } x, y \in I \text{ with } |x - y| < \delta.$$

But $|x_n - y_n| < \delta$, for $n > N$ say, by statement 1, so

$$|f(x_n) - f(y_n)| < \varepsilon, \quad \text{for } n > N,$$

contradicting statement 2. Thus f is not uniformly continuous on I. ■

Theorem F18 gives us the second part of the following strategy; the first part is an elaboration of the definition of uniform continuity.

Strategy F4

- To prove that a function f is uniformly continuous on an interval I, find an expression for $\delta > 0$ in terms of a given $\varepsilon > 0$ such that

  $$|f(x) - f(y)| < \varepsilon, \quad \text{for all } x, y \in I \text{ with } |x - y| < \delta.$$

- To prove that a function f is *not* uniformly continuous on an interval I, find two sequences (x_n) and (y_n) in I, and $\varepsilon > 0$, such that

  $$|x_n - y_n| \to 0 \text{ as } n \to \infty, \text{ and}$$
  $$|f(x_n) - f(y_n)| \geq \varepsilon, \quad \text{for } n = 1, 2, \ldots.$$

When using part 2 of Strategy F4, you should aim to choose the terms x_n and y_n close together at points of I where the graph of f is steep; see Exercise F13(b), for example.

We could have applied part 2 of Strategy F4 in Worked Exercise F8 by taking $x_n = 1/(2n)$ and $y_n = 1/n$, for $n = 1, 2, \ldots$, since (x_n) and (y_n) lie in I:

$$|x_n - y_n| = \left|\frac{1}{2n} - \frac{1}{n}\right| = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

and, for $n = 1, 2, \ldots$,

$$|f(x_n) - f(y_n)| = \left|\frac{1}{x_n} - \frac{1}{y_n}\right| = \left|\frac{1}{1/(2n)} - \frac{1}{1/n}\right| = n \geq 1.$$

Thus, by taking $\varepsilon = 1$ in part 2 of Strategy F4, we deduce that f is not uniformly continuous on I.

Try the next exercise using Strategy F4.

Exercise F13

(a) Prove that $f(x) = x^3$ is uniformly continuous on $I = [-2, 2]$.

(b) Prove that $f(x) = x^2$ is not uniformly continuous on $I = \mathbb{R}$.

Hint: In part (b), take $x_n = n + 1/n$ and $y_n = n$, for $n = 1, 2, \ldots$.

4.2 A condition that ensures uniform continuity

Checking uniform continuity from the definition can be complicated. However, we can often deduce uniform continuity in a straightforward way from the following fundamental result. Like the Intermediate Value Theorem and the Extreme Value Theorem that you met in Section 3 of Unit D4, this result is another illustration of the fact that continuous functions on bounded closed intervals have particularly nice properties.

Theorem F19

If the function f is continuous on a bounded closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

You saw in Worked Exercise F7 that the function $f(x) = x^2$ is uniformly continuous on $[-4, 4]$. In fact this can be deduced immediately from Theorem F19, as follows. The function $f(x) = x^2$ is continuous on the whole of $\mathbb{R}$, and hence on any bounded closed interval, so it must be uniformly continuous on *any* bounded closed interval, by Theorem F19. However, $f(x) = x^2$ is not *uniformly* continuous on the set $\mathbb{R}$, as you saw in Exercise F13(b): the image values of points near to a point c approach $f(c)$ more and more slowly as the point c gets larger and larger.

Exercise F14

Use Theorem F19 to prove that $f(x) = x^3$ is uniformly continuous on $I = [-2, 2]$.

Proof of Theorem F19 We assume that f is continuous on $[a, b]$ but *not* uniformly continuous on $[a, b]$, and deduce a contradiction using the bisection method.

You met the bisection method in the proofs of the Intermediate Value Theorem and the Extreme Value Theorem in Section 3 of Unit D4.

Since we are assuming that f is not uniformly continuous on $[a, b]$, it follows from Theorem F18 that there exist sequences (x_n) and (y_n) in $[a, b]$ and $\varepsilon > 0$, such that

$$|x_n - y_n| \to 0 \quad \text{as } n \to \infty, \text{ and} \tag{21}$$

$$|f(x_n) - f(y_n)| \geq \varepsilon, \quad \text{for } n = 1, 2, \ldots. \tag{22}$$

Let $a_0 = a$, $b_0 = b$ and $p = \frac{1}{2}(a_0 + b_0)$. Then at least one of $[a_0, p]$ or $[p, b_0]$ must contain terms x_n for infinitely many $n \in \mathbb{N}$. We denote this interval by $[a_1, b_1]$, choosing either interval if both contain infinitely many terms x_n. Thus we have:

1. $[a_1, b_1] \subseteq [a_0, b_0]$
2. $b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$
3. $[a_1, b_1]$ contains terms x_n for infinitely many $n \in \mathbb{N}$.

Now we repeat this process, bisecting $[a_1, b_1]$ to obtain $[a_2, b_2]$, and so on. This gives a sequence of closed intervals

$$[a_k, b_k], \quad k = 0, 1, 2, \ldots,$$

such that the following properties hold for $k = 0, 1, 2, \ldots$

1. $[a_{k+1}, b_{k+1}] \subseteq [a_k, b_k]$
2. $b_k - a_k = \left(\frac{1}{2}\right)^k (b_0 - a_0)$
3. $[a_k, b_k]$ contains terms x_n for infinitely many $n \in \mathbb{N}$.

We use k here to avoid n having two meanings.

Now property 1 implies that (a_k) is increasing and bounded above by b_0. Hence (a_k) is convergent by the Monotone Convergence Theorem (see Section 5 of Unit D2).

Moreover, if we let $\lim\limits_{k\to\infty} a_k = c$, then it follows that $\lim\limits_{k\to\infty} b_k = c$ also, by property 2 and the Combination Rules for sequences, and by property 3, the sequence (x_n) contains a subsequence (x_{n_k}) such that

$$x_{n_k} \in [a_k, b_k], \quad \text{for } k = 0, 1, 2, \ldots.$$

Thus $\lim\limits_{k\to\infty} x_{n_k} = c$, by the Squeeze Rule for sequences, so

$$y_{n_k} = (y_{n_k} - x_{n_k}) + x_{n_k} \to 0 + c = c,$$

by statement (21). It now follows from the continuity of f at the point c that

$$\lim_{k\to\infty} f(x_{n_k}) = f(c) \quad \text{and} \quad \lim_{k\to\infty} f(y_{n_k}) = f(c),$$

but this contradicts statement (22). Hence our original assumption that f is continuous but not uniformly continuous on $[a, b]$ must be false. This completes the proof. ■

As part of this proof, we showed that any sequence (x_n) in $[a, b]$ contains a convergent subsequence (x_{n_k}). This remarkable result, which is of importance in many parts of analysis, is called the *Bolzano–Weierstrass Theorem*. It can be stated as follows.

Theorem F20 Bolzano–Weierstrass Theorem

Any bounded sequence has a convergent subsequence.

Summary

In this unit you have studied the limiting behaviour of a real function f near a point c, using the concept of a punctured neighbourhood of a point c. You have seen that $f(x)$ has limit $f(c)$ as x tends to c precisely when f is continuous at the point c. You have also studied the limiting behaviour of f as x tends to infinity, and seen what it means for the function values $f(x)$ to tend to infinity as x tends to c. You have learnt how to use a variety of rules to help you analyse such asymptotic behaviour and to evaluate different limits.

You have also met a new definition of continuity: the ε-δ definition of continuity. You have seen that this is equivalent to the sequential definition of continuity that you studied in Unit D4 and, although it is more abstract than the sequential definition, it can be easier to use in certain situations. In particular, you have used both definitions to prove different properties of some interesting functions: the Dirichlet function, the Riemann function and the blancmange function.

Finally you have learnt what it means for a function to be *uniformly continuous* on an interval. You have seen how to give a direct proof that a function has this property and met the result that a function that is continuous on a bounded closed interval is uniformly continuous.

As you continue your studies, you will see that the ideas you have met in this unit play a key role in the foundations of calculus. In particular, limits are fundamental in the theories of differentiation and integration, and uniform continuity is used to show that a function that is continuous on a bounded closed interval can be integrated.

Learning outcomes

After working through this unit, you should be able to:

- understand the statement $\lim_{x\to c} f(x) = l$, or $f(x) \to l$ as $x \to c$
- appreciate the relationship between limits of a function and continuity
- use the Combination Rules, Composition Rule and Squeeze Rule to calculate limits of functions
- understand one-sided limits, and the statements $\lim_{x\to c^+} f(x) = l$ and $\lim_{x\to c^-} f(x) = l$
- use certain basic limits to find other limits
- understand the *asymptotic behaviour* of functions, and the statements $f(x) \to \infty$ as $x \to c$ (or c^+ or c^-) and $f(x) \to l$ as $x \to \infty$ (or $-\infty$)
- use the Reciprocal Rule, the Combination Rules, the Squeeze Rule and the Composition Rule to determine the asymptotic behaviour of functions
- understand and use the ε–δ definition of continuity
- describe functions which are highly discontinuous, or continuous but very irregular
- understand and use the ε–δ definition of a limit
- understand the definition of *uniform continuity* and use it in simple cases
- know and use various conditions for uniform continuity.

Solutions to exercises

Solution to Exercise F1

(a) The function

$$f(x) = \frac{x^2 + x}{x}$$

has domain $\mathbb{R} - \{0\}$, so f is defined on any punctured neighbourhood of 0. Also,

$$f(x) = \frac{x(x+1)}{x} = x + 1, \quad \text{for } x \neq 0.$$

Thus if (x_n) lies in $\mathbb{R} - \{0\}$ and $x_n \to 0$, then

$$f(x_n) = x_n + 1 \to 1.$$

Hence

$$\lim_{x \to 0} \frac{x^2 + x}{x} = 1.$$

(b) The domain of $f(x) = \lfloor x \rfloor$ is $\mathbb{R}$, so f is defined on any punctured neighbourhood of 1. Now,

$$f(x) = 0, \quad \text{for } 0 \leq x < 1,$$

and

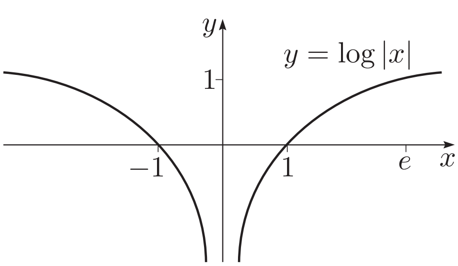
$$f(x) = 1, \quad \text{for } 1 \leq x < 2.$$

The two sequences $(1 + 1/n)$ and $(1 - 1/n)$ both tend to 1, and have terms lying in $\mathbb{R} - \{1\}$. Also,

$$\lim_{n \to \infty} f(1 + 1/n) = 1 \quad \text{but} \quad \lim_{n \to \infty} f(1 - 1/n) = 0.$$

Hence $\lim_{x \to 1} \lfloor x \rfloor$ does not exist, by the first part of Strategy F1.

(c) The function $f(x) = \log |x|$ has domain $\mathbb{R} - \{0\}$, so it is defined on any punctured neighbourhood of 0, and its graph is as follows. (This is included for interest – you are not expected to have sketched the graph as part of your solution.)



We consider the null sequence $x_n = 1/n$. Then

$$f(x_n) = \log |x_n| = \log(1/n) = -\log n.$$

Now $\log n \to \infty$ as $n \to \infty$, so $f(x_n) \to -\infty$ as $n \to \infty$.

Hence $\lim_{x \to 0} \log |x|$ does not exist, by the second part of Strategy F1.

Solution to Exercise F2

(a) The domain of the continuous function $f(x) = \sqrt{x}$ is the interval $[0, \infty)$, so f is defined on the open interval $(0, \infty)$, which contains 2. Hence, by Theorem F2, we have

$$\lim_{x \to 2} \sqrt{x} = \sqrt{2}.$$

(b) Since $\sin x > 0$ for $0 < x < \pi$, the function $f(x) = \sqrt{\sin x}$ is defined on the open interval $(0, \pi)$, which contains $\pi/2$, and is continuous, by the Composition Rule for continuous functions. Hence, by Theorem F2, we have

$$\lim_{x \to \pi/2} \sqrt{\sin x} = \sqrt{\sin(\pi/2)} = 1.$$

(c) The function $f(x) = e^x/(1 + x)$ is defined on the open interval $(-1, \infty)$, which contains 1, and is continuous, by the Quotient Rule for continuous functions. Hence, by Theorem F2, we have

$$\lim_{x \to 1} \frac{e^x}{1 + x} = \frac{e^1}{1 + 1} = \tfrac{1}{2}e.$$

Solution to Exercise F3

(a) First we write

$$\frac{\sin x}{2x + x^2} = \frac{\sin x}{x(2 + x)} = \left(\frac{\sin x}{x}\right)\left(\frac{1}{2 + x}\right).$$

Since

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \text{(by Theorem F1)}$$

and

$$\lim_{x \to 0} \frac{1}{2 + x} = \tfrac{1}{2},$$

because $\dfrac{1}{2 + x}$ is continuous at 0, we deduce by the

Product Rule for limits that

$$\lim_{x\to 0}\frac{\sin x}{2x+x^2}=1\times\tfrac{1}{2}=\tfrac{1}{2}.$$

(b) We can write

$$\frac{\sin(\sin x)}{\sin x}=g(f(x)),$$

where $f(x)=\sin x$ and $g(x)=\dfrac{\sin x}{x}$.

Substituting $u=f(x)=\sin x$ and using the fact that u is continuous at 0, we have

$$u=\sin x\to\sin 0=0\quad\text{as } x\to 0$$

and

$$g(u)=\frac{\sin u}{u}\to 1\quad\text{as } u\to 0.$$

The first proviso to the Composition Rule holds, because $f(x)=\sin x\neq 0$ in $N_1(0)$, for example.

Thus, by the Composition Rule,

$$g(f(x))=\frac{\sin(\sin x)}{\sin x}\to 1\quad\text{as } x\to 0.$$

(c) We can write

$$\left(\frac{x}{\sin x}\right)^{1/2}=g(f(x)),$$

where $f(x)=\dfrac{\sin x}{x}$ and $g(x)=1/x^{1/2}$.

Substituting $u=f(x)=\dfrac{\sin x}{x}$, we have

$$u=\frac{\sin x}{x}\to 1\quad\text{as } x\to 0,$$
$$g(u)=1/u^{1/2}\to 1/1^{1/2}=1\quad\text{as } u\to 1,$$

since g is continuous at 1, which also tells us that the second proviso to the Composition Rule holds.

Thus, by the Composition Rule,

$$g(f(x))=\left(\frac{x}{\sin x}\right)^{1/2}\to 1\quad\text{as } x\to 0.$$

(d) Using the hint, we obtain

$$\begin{aligned}\frac{1-\cos x}{x}&=\frac{2\sin^2\left(\frac{1}{2}x\right)}{x}\\&=\frac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x}\times\sin\left(\tfrac{1}{2}x\right).\end{aligned}$$

Now,

$$\lim_{x\to 0}\frac{\sin\left(\frac{1}{2}x\right)}{\frac{1}{2}x}=1,$$

by Worked Exercise F3(a), and

$$\lim_{x\to 0}\sin\left(\tfrac{1}{2}x\right)=0,$$

by the continuity of the function $x\longmapsto\sin\left(\frac{1}{2}x\right)$.

Thus, by the Product Rule,

$$\lim_{x\to 0}\frac{1-\cos x}{x}=1\times 0=0.$$

Solution to Exercise F4

(a) Since

$$1+x\le e^x\le\frac{1}{1-x},\qquad\text{for } |x|<1,$$

we have

$$x\le e^x-1\le\frac{1}{1-x}-1=\frac{x}{1-x},\qquad\text{for } |x|<1.$$

Thus if $0<x<1$, then

$$1\le\frac{e^x-1}{x}\le\frac{1}{1-x}=1+\frac{x}{1-x}=1+\frac{|x|}{1-x},$$

and if $-1<x<0$, then

$$1\ge\frac{e^x-1}{x}\ge\frac{1}{1-x}=1+\frac{x}{1-x}=1-\frac{|x|}{1-x}.$$

Since we also have

$$1-\frac{|x|}{1-x}<1\qquad\text{when } 0<x<1$$

and

$$1+\frac{|x|}{1-x}>1\qquad\text{when } -1<x<0,$$

we deduce that the inequalities

$$1-\frac{|x|}{1-x}\le\frac{e^x-1}{x}\le 1+\frac{|x|}{1-x}$$

hold whenever $0<|x|<1$, as required.

(b) We have

$$\lim_{x\to 0}\left(1-\frac{|x|}{1-x}\right)=1$$

and

$$\lim_{x\to 0}\left(1+\frac{|x|}{1-x}\right)=1,$$

because both functions are continuous on the interval $(-1, 1)$.

Thus, by part (a) and the Squeeze Rule,

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Solution to Exercise F5

(a) Since $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ and the function $x \longmapsto \sqrt{x}$ is continuous at 0, we have

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1 \quad \text{(by Theorem F7)}$$

and

$$\lim_{x \to 0^+} \sqrt{x} = 0 \quad \text{(by Theorem F8).}$$

Hence, by the Sum Rule,

$$\lim_{x \to 0^+} \left(\frac{\sin x}{x} + \sqrt{x} \right) = 1 + 0 = 1.$$

(b) We can write

$$\frac{\sin \sqrt{x}}{\sqrt{x}} = g(f(x)),$$

where $f(x) = \sqrt{x}$ and $g(x) = \dfrac{\sin x}{x}$.

Substituting $u = f(x) = \sqrt{x}$, we have

$$u = \sqrt{x} \to 0 \text{ as } x \to 0^+,$$

$$g(u) = \frac{\sin u}{u} \to 1 \text{ as } u \to 0,$$

so that

$$\lim_{u \to 0^+} \frac{\sin u}{u} = 1 \quad \text{(by Theorem F7).}$$

Moreover, $f(x) = \sqrt{x} \neq 0$ on the open interval $(0, 1)$, so the first proviso of the one-sided limit version of the Composition Rule holds.

Thus, by the Composition Rule,

$$g(f(x)) = \frac{\sin \sqrt{x}}{\sqrt{x}} \to 1 \text{ as } x \to 0^+.$$

Solution to Exercise F6

(a) Let $f(x) = |x|$; then $f(x) > 0$ for $x \in \mathbb{R} - \{0\}$, and

$$\lim_{x \to 0} |x| = 0,$$

since f is continuous at 0. Hence, by the Reciprocal Rule,

$$\frac{1}{f(x)} = \frac{1}{|x|} \to \infty \text{ as } x \to 0.$$

(b) Let $f(x) = x^3 / \sin x$; then

$$f(x) = \frac{x^2}{(\sin x)/x} > 0, \quad \text{for } x \in N_\pi(0),$$

and

$$f(x) = \frac{x^2}{(\sin x)/x} \to \frac{0}{1} = 0 \text{ as } x \to 0,$$

by Theorem F6(a) and the Quotient Rule, since the function $x \longmapsto x^2$ is continuous at 0.

Hence, by the Reciprocal Rule,

$$\frac{1}{f(x)} = \frac{\sin x}{x^3} \to \infty \text{ as } x \to 0.$$

(c) Let $f(x) = x^3 - 1$; then $f(x) > 0$ for $x \in (1, \infty)$, and

$$\lim_{x \to 1^+} x^3 - 1 = 0,$$

since f is continuous at 1. Hence, by the Reciprocal Rule,

$$\frac{1}{f(x)} = \frac{1}{x^3 - 1} \to \infty \text{ as } x \to 1^+.$$

Solution to Exercise F7

(a) Since

$$f(x) = \frac{2x^3 + x}{x^3} = 2 + \frac{1}{x^2}, \quad \text{for } x \neq 0,$$

we deduce, by Theorem F11(b) and the Sum Rule, that

$$f(x) = 2 + \frac{1}{x^2} \to 2 + 0 = 2 \text{ as } x \to \infty.$$

(b) Let

$$f(x) = \frac{x^2}{2x^3 + 1};$$

then dividing both the numerator and the denominator by the dominant term x^3, we obtain

$$f(x) = \frac{1/x}{2 + 1/x^3} \to \frac{0}{2 + 0} = 0 \text{ as } x \to \infty,$$

by Theorem F11(b) and the Combination Rules.

Hence, by the Reciprocal Rule,

$$\frac{1}{f(x)} = \frac{2x^3+1}{x^2} \to \infty \quad \text{as } x \to \infty.$$

Solution to Exercise F8

Since $-1 \leq \sin(1/x) \leq 1$ for $x \neq 0$, we have

$$-\frac{1}{x} \leq \frac{\sin(1/x)}{x} \leq \frac{1}{x}, \quad \text{for } x \in (0,\infty).$$

Also, by Theorem F11(b) and the Multiple Rule,

$$g(x) = -\frac{1}{x} \to 0 \quad \text{as } x \to \infty$$

and

$$h(x) = \frac{1}{x} \to 0 \quad \text{as } x \to \infty.$$

Thus, by the Squeeze Rule, part (a),

$$f(x) = \frac{\sin(1/x)}{x} \to 0 \quad \text{as } x \to \infty.$$

Solution to Exercise F9

(a) By Theorem F13(b), we have

$$\frac{e^x}{x^2} \to \infty \quad \text{as } x \to \infty.$$

By Theorem F13(c), we have

$$\frac{\log x}{x^2} \to 0 \quad \text{as } x \to \infty.$$

Also, $(\log x)/x^2 > 0$ for $x > 1$, so

$$\frac{x^2}{\log x} \to \infty \quad \text{as } x \to \infty,$$

by the Reciprocal Rule.

Hence, by the Combination Rules,

$$\frac{e^x}{x^2} + \frac{3x^2}{\log x} \to \infty \quad \text{as } x \to \infty.$$

(b) Following the hint, we write

$$\frac{\log x}{e^x} = \left(\frac{\log x}{x}\right)\left(\frac{x}{e^x}\right).$$

By Theorem F13(b) and (c),

$$\frac{x}{e^x} \to 0 \quad \text{as } x \to \infty \quad \text{and} \quad \frac{\log x}{x} \to 0 \quad \text{as } x \to \infty.$$

Thus, by the Product Rule,

$$\frac{\log x}{e^x} \to 0 \times 0 = 0 \quad \text{as } x \to \infty.$$

(c) The dominant term is e^x, so we write

$$\frac{2e^x - x^2}{e^x + \log x} = \frac{2 - x^2/e^x}{1 + (\log x)/e^x}.$$

Thus, by part (b) above, Theorem F13(b) and the Combination Rules,

$$\frac{2e^x - x^2}{e^x + \log x} \to \frac{2-0}{1+0} = 2 \quad \text{as } x \to \infty.$$

Solution to Exercise F10

(a) We can write

$$\frac{e^{x^2}}{x^2} = g(f(x)),$$

where $f(x) = x^2$ and $g(x) = e^x/x$.

Substituting $u = f(x) = x^2$, we have (by Theorem F13(a) and (b))

$$u = x^2 \to \infty \quad \text{as } x \to \infty,$$

$$g(u) = \frac{e^u}{u} \to \infty \quad \text{as } u \to \infty.$$

Thus, by the Composition Rule,

$$g(f(x)) = \frac{e^{x^2}}{x^2} \to \infty \quad \text{as } x \to \infty.$$

(b) We can write

$$\log(\log x) = g(f(x)),$$

where $f(x) = \log x$ and $g(x) = \log x$.

Substituting $u = f(x) = \log x$, we have (by Theorem F13(c))

$$u = \log x \to \infty \quad \text{as } x \to \infty,$$

$$g(u) = \log u \to \infty \quad \text{as } u \to \infty.$$

Thus, by the Composition Rule,

$$g(f(x)) = \log(\log x) \to \infty \quad \text{as } x \to \infty.$$

(c) We can write

$$x\sin(1/x) = \frac{\sin(1/x)}{1/x} = g(f(x)),$$

where $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{\sin x}{x}$.

Substituting $u = f(x) = 1/x$, we have

$$u = 1/x \to 0 \quad \text{as } x \to \infty \quad \text{(by Theorem F11(b))},$$

$$g(u) = \frac{\sin u}{u} \to 1 \quad \text{as } u \to 0 \quad \text{(by Theorem F1)}.$$

Thus, by the Composition Rule,

$$g(f(x)) = x\sin(1/x) \to 1 \text{ as } x \to \infty.$$

Solution to Exercise F11

The domain of $f(x) = x^3$ is $\mathbb{R}$.

Let $\varepsilon > 0$ be given. We want to choose $\delta > 0$, in terms of ε, such that

$$|f(x) - f(1)| < \varepsilon, \quad \text{for all } x \text{ with } |x-1| < \delta. \quad (*)$$

We follow the steps in Strategy F3.

1. First we write

$$f(x) - f(1) = x^3 - 1 = (x-1)(x^2+x+1).$$

2. Next we obtain an upper bound for $|x^2+x+1|$ when x is near 1. If $|x-1| \le 1$, then x lies in the interval $[0,2]$, so (by the Triangle Inequality)

$$\begin{aligned}|x^2+x+1| &\le |x|^2 + |x| + 1\\ &\le 2^2 + 2 + 1 = 7.\end{aligned}$$

3. Hence

$$|f(x) - f(1)| \le 7|x-1|, \quad \text{for } |x-1| \le 1.$$

So if $|x-1| < \delta$, where $0 < \delta \le 1$, then

$$|f(x) - f(1)| < 7\delta.$$

Thus, if we choose $\delta = \min\{1, \tfrac{1}{7}\varepsilon\}$, then

$$|f(x) - f(1)| < 7\delta \le \varepsilon,$$
for all x with $|x-1| < \delta$,

which proves statement $(*)$.

Thus f is continuous at the point 1.

Solution to Exercise F12

The domain of

$$f(x) = \frac{2x^3+3x-5}{x-1}$$

is $\mathbb{R} - \{1\}$, so f is defined on each punctured neighbourhood of 1. Also, for $x \ne 1$,

$$\begin{aligned}f(x) = \frac{2x^3+3x-5}{x-1} &= \frac{(x-1)(2x^2+2x+5)}{(x-1)}\\ &= 2x^2+2x+5.\end{aligned}$$

This suggests that

$$\lim_{x\to 1} f(x) = 2\times 1^2 + 2\times 1 + 5 = 9,$$

so we must prove that

for each $\varepsilon > 0$, there exists $\delta > 0$ such that
$|f(x) - 9| < \varepsilon$, for all x with $0 < |x-1| < \delta$. $(*)$

1. First we write, for $x \ne 1$,

$$\begin{aligned}f(x) - 9 &= 2x^2 + 2x + 5 - 9\\ &= 2(x^2 + x - 2)\\ &= 2(x-1)(x+2).\end{aligned}$$

2. Next, if $|x-1| \le 1$, then x lies in the interval $[0,2]$, so (by the Triangle Inequality)

$$|2(x+2)| \le 2(|x|+2) \le 2(2+2) = 8.$$

3. Hence

$$|f(x) - 9| \le 8|x-1|, \quad \text{for } 0 < |x-1| \le 1.$$

So if $0 < |x-1| < \delta$, where $0 < \delta \le 1$, then

$$|f(x) - 9| < 8\delta.$$

Thus, if we choose $\delta = \min\{1, \tfrac{1}{8}\varepsilon\}$, then

$$|f(x) - 9| < 8\delta \le \varepsilon,$$

for all x with $0 < |x-1| < \delta$,

which proves statement $(*)$.

Hence

$$\lim_{x\to 1} f(x) = 9.$$

Solution to Exercise F13

(a) We use part 1 of Strategy F4. Let $\varepsilon > 0$ be given. We have

$$\begin{aligned}f(x) - f(y) &= x^3 - y^3\\ &= (x-y)(x^2+xy+y^2),\end{aligned}$$

so for $x, y \in [-2,2]$ (by the Triangle Inequality),

$$\begin{aligned}|f(x) - f(y)| &= |x-y|\,|x^2+xy+y^2|\\ &\le (|x|^2 + |x|\,|y| + |y|^2)|x-y|\\ &\le 12|x-y|,\end{aligned}$$

since $|x| \le 2$ and $|y| \le 2$.

Thus, if we choose $\delta = \frac{1}{12}\varepsilon$, then whenever $x, y \in [-2,2]$ and $|x-y| < \delta$, we have

$$|f(x) - f(y)| \le 12|x-y| < 12 \times \tfrac{1}{12}\varepsilon = \varepsilon.$$

Hence f is uniformly continuous on $[-2,2]$.

(b) Following part 2 of Strategy F4 and the hint, we take $x_n = n + 1/n$ and $y_n = n$, for $n = 1, 2, \ldots$. Both sequences lie in $I = \mathbb{R}$ and

$$\begin{aligned}|x_n - y_n| &= |(n + 1/n) - n| \\ &= 1/n \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

$$\begin{aligned}|f(x_n) - f(y_n)| &= |x_n^2 - y_n^2| \\ &= (n + 1/n)^2 - n^2 \\ &= n^2 + 2n(1/n) + (1/n)^2 - n^2 \\ &= 2 + 1/n^2 \geq 2, \quad \text{for } n = 1, 2, \ldots.\end{aligned}$$

Thus, by taking $\varepsilon = 2$ in part 2 of Strategy F4, we deduce that f is not uniformly continuous on $\mathbb{R}$.

Solution to Exercise F14

Since $f(x) = x^3$ is continuous on its domain $\mathbb{R}$, we deduce that f is continuous on the bounded closed interval $[-2, 2]$. Thus f is uniformly continuous on $[-2, 2]$, by Theorem F19.